N1 • Ordering fractions and decimals

Mathematical goals

To help learners to:

- interpret decimals and fractions using scales and areas;
- find equivalent fractions;
- relate fractions and decimals;
- order fractions and decimals;

and to reflect on and discuss these processes.

These goals may be adapted for learners aiming at lower level qualifications. For example, you may decide to focus just on interpreting and ordering decimals.

Starting points

Learners will have met these concepts before. Many, however, may still have misconceptions and difficulties. Typically, these include:

- confusing decimal and fraction notation (e.g. $\frac{1}{4}$ is confused with 1.4 or 0.4);
- believing that the magnitude of a decimal depends on the number of digits it contains (e.g. $0.62 > 0.8$ because $62 > 8$, or $0.4 < 0.32$ because 0.4 is in tenths and 0.32 is in hundredths);
- ignoring numerators when comparing fractions (e.g. $\frac{1}{4} > \frac{3}{5}$ because quarters are greater than fifths).

During the session, learners will confront and discuss such misconceptions. Also, many learners associate fractions with part/whole areas and decimals with number lines. This activity aims to build a more connected understanding of all these ideas.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Decimals*;
- Card set B – *Fractions*;
- Card set C – *Areas*;
- Card set D – *Scales*.

For learners who struggle, keep in reserve Card set E – *Areas* and Card set F – *Scales*.

For learners aiming at lower levels, use Card sets E (instead of C) and F (instead of D).

Each card set includes one blank for learners to use when creating their own cards.

Time needed

Between 30 minutes and 1 hour, though this will vary depending on the learners involved.

Suggested approach

For learners aiming at lower levels, you could just use question 1.

For learners aiming at lower levels, you could just use Card sets A, E and F.

Beginning the session

Write the following two questions on the board:

1. Write down these decimals in order of size, from smallest to largest. Underneath, describe and explain your method for doing this.

   0.75 0.04 0.375 0.25 0.4 0.125 0.8

2. Write down these fractions in order of size from smallest to largest. Again, describe and explain your method.

   $\frac{3}{4}$ $\frac{3}{8}$ $\frac{2}{5}$ $\frac{8}{10}$ $\frac{1}{4}$ $\frac{1}{25}$ $\frac{1}{8}$

Ask learners to write down their answers and methods, on their own, without discussion. Allow about five minutes for this. The intention is to expose learners' existing interpretations and misconceptions, not to put them right. The rest of the session should enable learners to answer the questions correctly. Sometimes, surprising responses are revealed:

e.g. 0.375, 0.125, 0.75, 0.25, 0.04, 0.4, 0.8
I know this because they work like fractions, 0.4 is like a quarter. The more digits there are, the smaller the decimal is.

e.g. 0.125, 0.375, 0.04, 0.25, 0.75, 0.4, 0.8
Tenths are bigger than hundredths and thousandths, so longer decimals are smaller.

Working in groups

Ask learners to sit in pairs or threes (with learners who disagree sitting next to each other to encourage more discussion) and give each group of learners Card sets A, B, C and D.

Ask learners to take it in turns to match pairs of cards and place them on the table, side by side (not on top of one another, or later pairings will be more difficult). As they do this they must explain how they know that the cards make a pair. When they have given their explanation, their partner(s) should either challenge what they have said or say why they agree.

As you go round the room, listen to learners' explanations. Note down any obvious misconceptions that emerge, for whole group discussion at the end of the session. Encourage learners to explain carefully why pairs of cards match each other.

If you see learners in difficulty, give them Card sets E and F. These have areas and scales divided into hundredths and tenths. Learners may find these easier to match. When they have done this, remove Card sets E and F and ask learners to return to the original challenge.

When learners are happy with their final matching, ask them to place the cards in order of size, smallest to largest.

Learners who find the activity straightforward may be asked to make new cards that fit between the pairs of cards they have put in order.

When learners have completed the activity, ask them to revisit the answers they wrote down at the beginning of the lesson. Did they make any mistakes? Encourage them to write down and explain any faults in their initial reasoning.

Reviewing and extending learning

Hold a whole group discussion about what has been learned, drawing out common misconceptions and discussing them explicitly.

Follow up with questions using mini-whiteboards. Choose questions carefully, so that they range in difficulty. Target them at individuals at an appropriately challenging level. For example:

You could ask learners aiming at lower levels to continue decimal sequences, such as:
0.2, 0.4, 0.6, . . .
0.3, 0.6, 0.9, . . .

Show me . . .

- a number between 0.5 and 0.6; between 0.5 and 0.51 . . .
- a fraction between $\frac{1}{8}$ and $\frac{1}{4}$; $\frac{2}{5}$ and $\frac{3}{5}$; $\frac{1}{3}$ and $\frac{2}{7}$ . . .
- a fraction equal to 0.1, 0.2, 0.3, 0.4, . . . ; 0.25, 0.5, 0.75 . . .
- a decimal equal to $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, . . . ; $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, . . .
- an area diagram showing $\frac{1}{2} + \frac{1}{3}$; now a number line diagram . . .

Ask learners to make notes on what they have learned and on how they felt about learning in this way.

What learners might do next

Learners could play 'guess my number'. One learner thinks of a number and the other learners in the group have to guess what it is. After each guess, the first learner replies 'too big' or 'too small'. Start with whole numbers. Later, try decimals, fractions and negative numbers.

Further ideas

This type of activity may be used in any topic where a range of representations is used. Examples in this pack include:

N5 Understanding the laws of arithmetic;

A1 Interpreting algebraic expressions;

SS6 Representing 3D shapes.

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N1 Card set A – *Decimals*

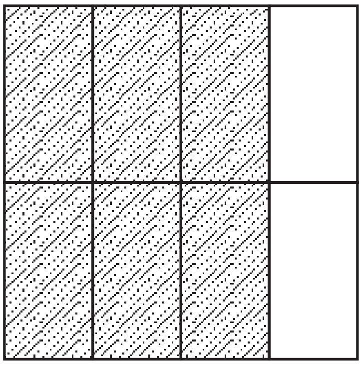
0.8	**0.04**
0.25	**0.375**
0.4	**0.125**
0.75	

N1 Card set B – *Fractions*

$\frac{3}{8}$	$\frac{2}{5}$
$\frac{1}{4}$	$\frac{3}{4}$
$\frac{8}{10}$	$\frac{1}{25}$
$\frac{1}{8}$	

N1 Card set C – *Areas*

C1



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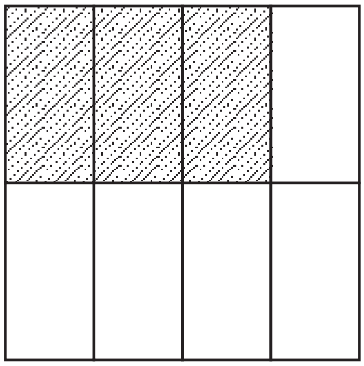

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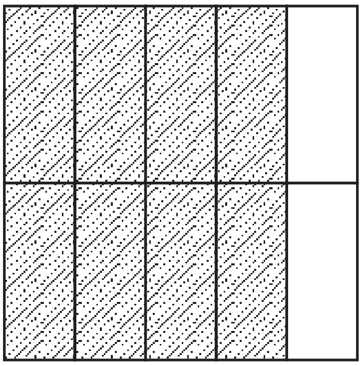

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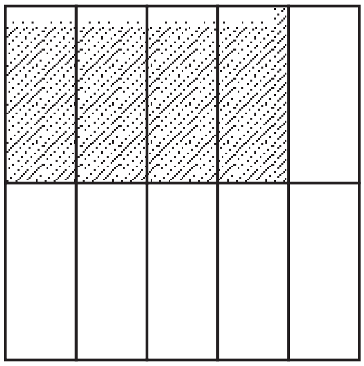

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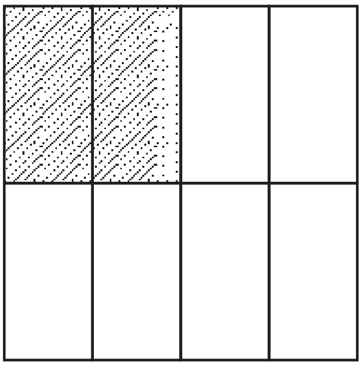

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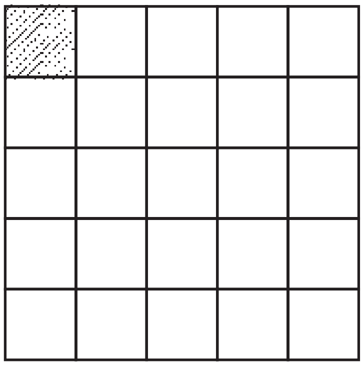
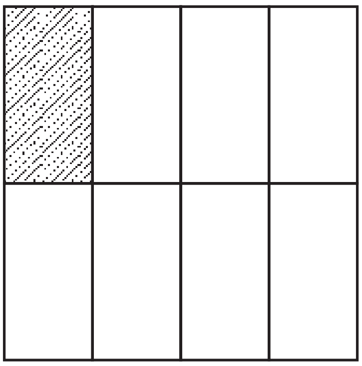


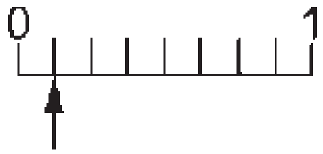
C7



C8

N1 Card set D – *Scales*

D1



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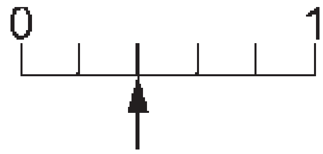

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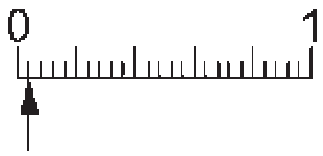

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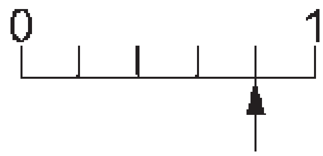

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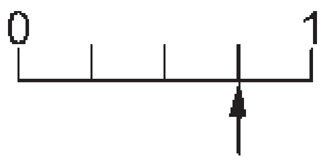

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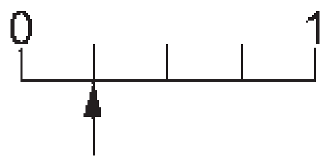

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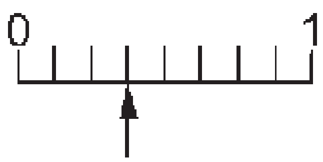
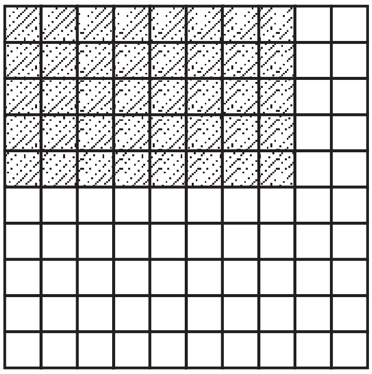


D8

N1 Card set E – *Areas*

E1



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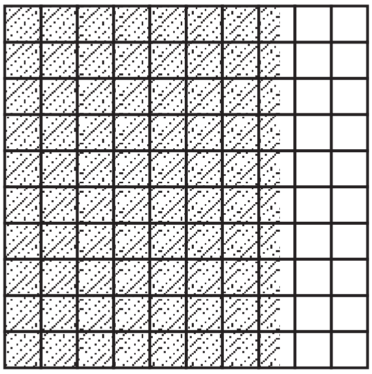

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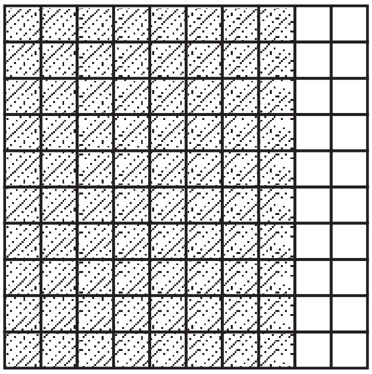

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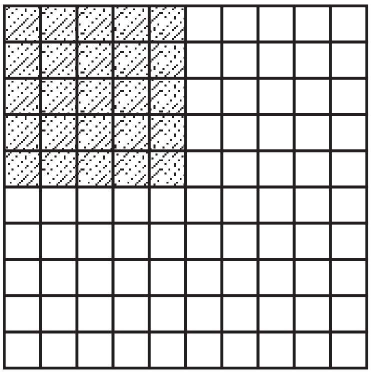

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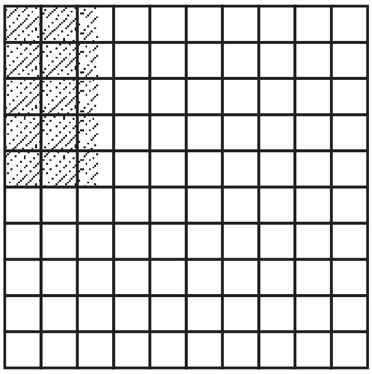

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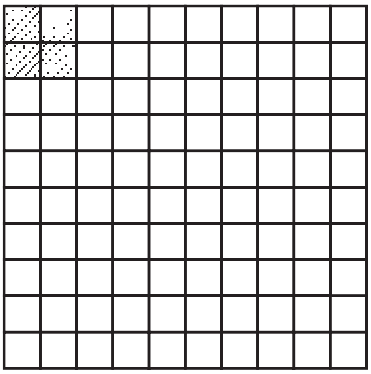

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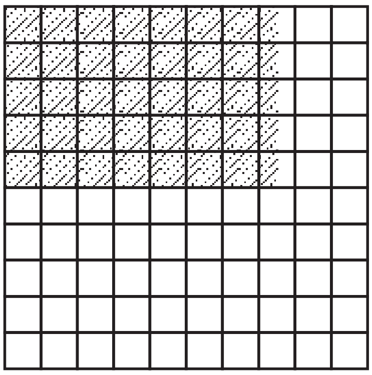
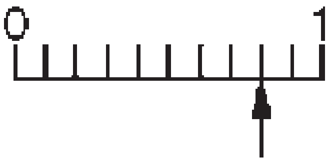


E8

N1 Card set F – *Scales*

F1



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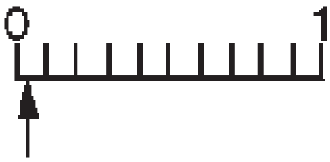

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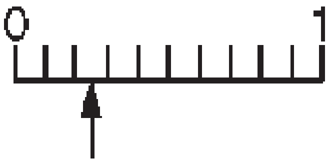

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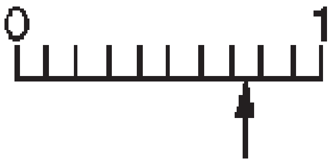

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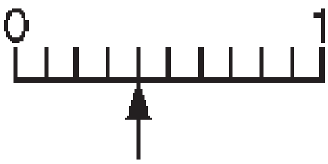

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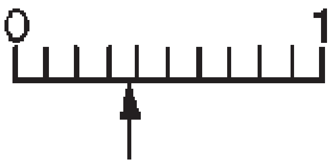

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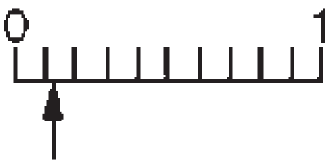


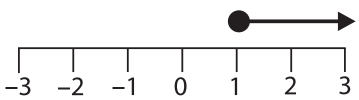
F8

N2 • Evaluating statements about number operations

Mathematical goals

To enable learners to:

- understand the properties of number operations;
- substitute integers, fractions, decimals and negative numbers into statements and equations in order to test their validity;
- address common misconceptions about the effect of addition, subtraction, multiplication, division, squaring and finding square roots.



Some learners may also begin to work with number line notation such as the example (left) representing $x \geq 1$.

Starting points

Although learners will be familiar with number operations, many will still have problems interpreting and using these concepts. This session is designed to expose and work on difficulties that learners have with:

- adding and subtracting negative numbers;
- interpreting division notation (e.g. misreading $x \div y$ as "how many xs go into y?");
- the commutativity of multiplication but not of division;
- the effect of dividing a smaller number by a larger number;
- the effect of multiplying and dividing by numbers less than 1.

Materials required

For each learner you will need:

- Sheet 1 – *Number operations*.

For each small group of learners you will need:

- Card set A – *Statements*;
- Card set B – *Number lines*;
- large sheet of paper for making a poster;
- glue stick;
- felt tip pen.

Calculators may be helpful for learners to check their answers.

Time needed

At least 1 hour. The exact time depends on whether or not you choose to use Card set B – *Number lines*.

Suggested approach

Beginning the session

As learners arrive, ask them to complete Sheet 1 – *Number operations*, working on their own. Learners frequently comment that it looks easy, and complete it in just a few minutes. It should, however, expose many misconceptions that can be discussed later in the session. Do not go through the answers at this stage. Tell learners to put the sheet to one side, face down, for the time being.

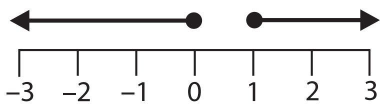
Write one of the harder statements from Card set A – *Statements* on the board e.g. $x^2 \geq x$. Ask learners to interpret this statement in words and to say whether they think it is a true statement or not. Typically, learners begin by saying that this is clearly true because, when you multiply a number by itself, it gets bigger. Ask questions about the statement:

Can you give me a value for x that makes the statement true?
Can you give me another? And another?
Try a fraction, a decimal, a negative number . . .

Can you give me a value for x that makes the statement false?
Can you give me another? And another?
Try a fraction, a decimal, a negative number . . .

Can we state precisely when the statement is true and when it is not?

Demonstrate how the result may be shown on a number line:



Explain the notation, where solid blobs denote that you do include the number and hollow blobs denote that you do not.

Explain that in this session learners will be asked to consider a number of statements in a similar way. Explain that the objective of the session is for each small group of learners to produce a poster which shows each statement classified according to whether it is always, sometimes or never true and furthermore:

- if it is sometimes true, then to write examples around the statement to show when it is true and when it is not true;
- if it is always true, then to give a variety of examples demonstrating that it is true, using large numbers, decimals, fractions and negative numbers if possible;
- if it is never true, then to write an explanation of how you can be sure that this is the case.

Working in groups

Ask learners to work in groups of two or three.

Give each group Card set A – *Statements*, a large sheet of paper, a glue stick and a felt tip pen.

Ask learners to divide their sheet into three columns and to head the columns with the words: 'Always True', 'Sometimes True', 'Never True'.

Learners now take it in turns to place a card in one of the columns and justify their answer to their partner(s). Their partner(s) must challenge them if the explanation has not been clear and complete. When the pair or group agrees, they should paste the card down and write examples around it to justify their choice. This should include examples and counter-examples. Learners should not need to rearrange the equations. Trial-and-error substitutions should be enough in most cases.

Learners who struggle should be given calculators to help with the arithmetic. Suggesting numbers for learners to substitute may help to take their thinking forward.

Learners who need an extra challenge should be encouraged to match the *Number lines* cards (Card set B) to the *Statements* cards (Card set A). For even greater challenge, you may wish to add further, more demanding, cards such as $x^2 + 4 = 13$.

When doing this activity, you may find that learners sort their cards quickly and superficially to begin with. They may need prompting to try fraction, decimal and negative substitutions to check their assumptions, if they do not do this of their own accord. Look for common misconceptions that surface and note these down for later discussion with the whole group.

Reviewing and extending learning

Ask learners to display their posters to the whole group and to describe one thing they have learned.

Name particular misconceptions you have identified as learners were working on the activity. For example:

- division is commutative ($10 \div x = x \div 10$);
- you cannot divide smaller numbers by larger ones;
- addition/multiplication/squaring always makes numbers bigger;
- subtraction/division/square rooting always makes numbers smaller;

. . . and so on.

Finally, ask learners to look again at the questions they attempted at the beginning of the session. They should correct any that they now know are incorrect and write down what they have learned in the space underneath.

What learners might do next

A logical follow-up to this session is **N5 Understanding the laws of arithmetic**, which focuses on the order in which number operations are carried out.

Further ideas

This activity is about examining a mathematical statement and deciding on its truth or falsehood. This idea may be used in many other topics and levels. Examples in this pack include:

- **A4 Evaluating algebraic expressions;**
- **SS4 Evaluating statements about length and area;**
- **S2 Evaluating probability statements.**

N2 Sheet 1 – *Number operations*

Write down the missing numbers in the following.

You will have another go at this sheet at the end of the session. This will help you to identify what you have learned.

1. $12 + 6 = \ldots$

   $6 + 12 = \ldots$

2. $12 - 6 = \ldots$

   $6 - 12 = \ldots$

3. $12 \times 6 = \ldots$

   $6 \times 12 = \ldots$

4. $12 \div 6 = \ldots$

   $6 \div 12 = \ldots$

5. $12 + \ldots = 24$

   $12 - \ldots = 24$

6. $12 + \ldots = 6$

   $12 - \ldots = 6$

7. $12 \times \ldots = 24$

   $12 \div \ldots = 24$

8. $12 \times \ldots = 6$

   $12 \div \ldots = 6$

At the end of the session, write down what you have learned in the space below.

If you still have questions or difficulties, write these down too.

N2 Card set A – *Statements*

$3 + x = x + 3$ It doesn't matter which way round you add, you get the same answer.	$2 - x = x - 2$ It doesn't matter which way round you subtract, you get the same answer.
$5 \times x = x \times 5$ It doesn't matter which way round you multiply, you get the same answer.	$x \div 2 = 2 \div x$ It doesn't matter which way round you divide, you get the same answer.
$5 + x > 5$ If you add a number to 5, your answer will be more than 5.	$x + 8 > x$ If you add 8 to a number, your answer will be more than the number.
$5 - x \leq 5$ If you take a number away from 5, your answer will be less than or equal to 5.	$x - 10 > x$ If you take 10 away from a number, the answer will be greater than the number.
$4x \geq 4$ If you multiply 4 by a number, your answer will be greater than or equal to 4.	$10x \geq x$ If you multiply 10 by a number, your answer will be greater than or equal to the number.
$\frac{x}{2} < x$ If you divide a number by 2, the answer will be less than the number.	$\frac{10}{x} \leq 10$ If you divide 10 by a number, your answer will be less than or equal to 10.
$\sqrt{x} \leq x$ The square root of a number is less than or equal to the number.	$x^2 \geq x$ The square of a number is greater than or equal to the number.

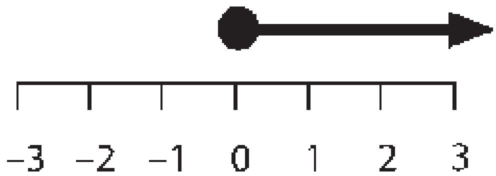
N2 Card set B – *Number lines*

B1

Sometimes true

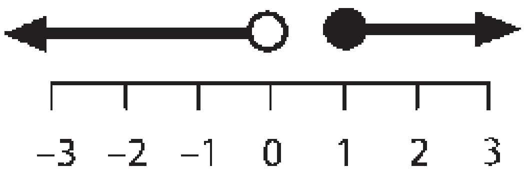
B2

Sometimes true



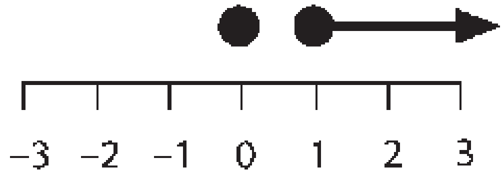
B3

Sometimes true



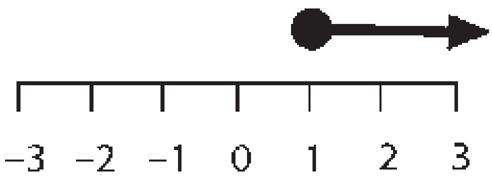
B4

Sometimes true



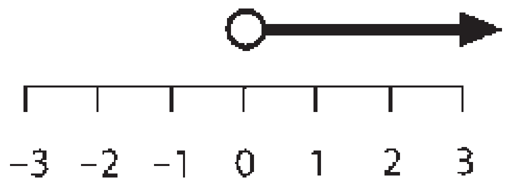
B5

Sometimes true



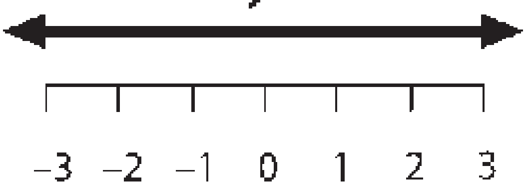
B6

Sometimes true



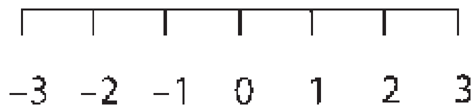
B7

Always true



B8

Never true



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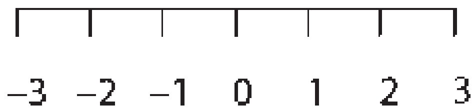
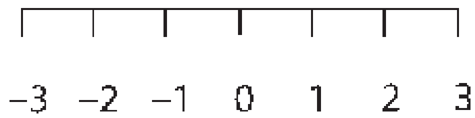


B10



N3 • Rounding numbers

Mathematical goals

To enable learners to:

- round numbers to the nearest 10;
- round numbers to the nearest 100;
- round numbers to the nearest 1000.

Starting points

Learners should have some awareness of place value.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- at least three one metre long strips of paper or card;
- felt tip pens;
- lots of blank cards approximately 2 cm square;
- glue stick.

Time needed

At least 30 minutes.

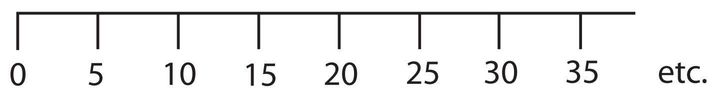
Suggested approach

Beginning the session

Give or ask for some examples of where an exact number is not needed and is sometimes not possible to determine, e.g. the distance travelled between two towns. Discuss why a number rounded to the nearest 10, 100 or 1 000 is sometimes enough, e.g. the size of a crowd at a demonstration.

Working in groups

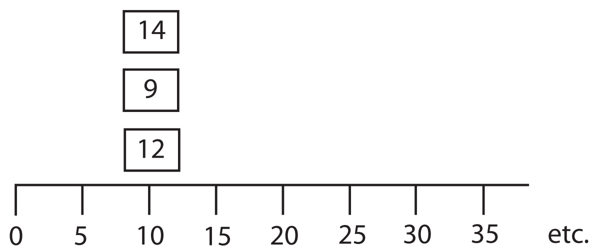
Ask learners to work in pairs and give each pair of learners a long strip of paper (or card). Ask them to mark it out, along the bottom edge, in 5s from 1 to 200 (or more if wanted).



The line does not have to be exactly to scale but the markings should be roughly evenly spaced.

Ask learners to write any number between 0 and 200 on a blank card and place it approximately where it fits on the number line. They should then move the card to the nearest 10 and stick it down. They should repeat this for at least 15 more numbers.

Numbers that are rounded to the same 10 should be stuck down above each other, making a column.



You may prefer to write a selection of suitable numbers on the board for learners to start with, before they think of their own numbers. Include consideration of numbers that are multiples of 5 and introduce the convention of rounding up.

Repeat the exercise for rounding to the nearest 100, using a strip marked in 50s from 1 to 2 000.

Repeat for rounding to the nearest 1 000, using a strip marked in 500s from 0 to 20 000.

Whole group discussion

Discuss how to round a number without using a number line. If necessary, practise some examples, using mini-whiteboards.

Reviewing and extending the learning

Using mini-whiteboards, ask questions such as:

> Give me a number that is 50 when rounded to the nearest 10.
>
> Give me a number that is 400 when rounded to the nearest 100.
>
> Give me a number that is 6 000 when rounded to the nearest 1 000.
>
> What is the smallest number that is rounded to 60 when rounded to the nearest 10?
>
> What is the largest number that is rounded to 700 when rounded to the nearest 100?

and so on.

What learners might do next

Round numbers to a given number of significant figures.

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N4 • Estimating length, using standard form

Mathematical goals

To help learners to:

- interpret decimals using metric units;
- estimate lengths;
- interpret standard form;

and to discuss and understand these processes.

Starting points

Some learners will have encountered decimals and standard form before. The opening discussion is used to recall these ideas.

Materials required

For each pair of learners you will need:

- Card set A – *Objects*;
- Card set B – *Measurements*;
- Card set C – *Measurements in standard form*;
- Card set D – *Comparisons*;
- calculator;

and optionally:

- glue stick;
- felt tip pen;
- large sheet of paper for making a poster.

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Explain that the distance from your nose to your finger tip is about one metre. Record this fact in the centre of the board. Ask learners to name objects or everyday distances that have lengths that are approximately 10 m, 100 m, 1 000 m, 10 000 m and then 0.1 m, 0.01 m, 0.001 m and 0.0001 m, using questions such as the following:

What is about 10 times as long as this?
(The distance across this room.)
What is 10 times the distance across the room?
(A sprint.)
What is 10 times the length of a sprint?
(Just over half a mile.)
What is one tenth the distance from nose to finger tip?
(The width of your hand.)
What is one tenth the width of your hand?
(The width of your little finger.)

List learners' answers on the board. This produces a table like this:

Metres	Rough size	Standard form
10 000	about 6 miles	10^4 m
1000	just over $\frac{1}{2}$ a mile	10^3 m
100	sprint	10^2 m
10	distance across this room	10^1 m
1	distance from nose to finger tip	10^0 m
0.1	width of hand	10^{-1} m
0.01	width of little finger	10^{-2} m
0.001	diameter of this blob •	10^{-3} m
0.0001	hair's breadth	10^{-4} m

Explain that, as we move up and down the list, we are multiplying and dividing lengths by ten. Introduce the standard form notation at this point.

Discuss the relationships between the lengths of objects in the list:

How long is the room in hand widths?

How far is 6 miles in hand widths?

Explain the relative significance of the decimal places:

Sarah is 1.6321 m tall. Is this a reasonable statement? Why?

Think of 1.6321 m as "One nose to finger tip + 6 hand widths + 3 finger widths + 2 full stops + 1 hair's breadth".

How would this number change if she put high heels on?
. . . if she flattened her hair slightly?
. . . if she sat down?

The examples may now be used to estimate the lengths of other everyday objects. Ask learners to name objects that are, for example, 0.02 m long (about two finger widths) or 0.005 m long (about the length of five full-stops placed side by side) and so on.

Working in groups

Give each pair of learners Card set A – *Objects* and Card set B – *Measurements*. Ask learners to match the objects to the corresponding measurements. If learners get stuck, suggest that they first arrange the objects in order of size.

Learners who struggle may find it helpful to work with a smaller set of cards, omitting those that show the greatest and smallest distances.

Six cards in Card set B have been left blank. It is intended that learners should use these to write their own estimates for the objects in Card set A with which they will be more familiar: the length of a fly/stapler/telephone/truck, the height of a desk, the wingspan of an aircraft.

When learners have completed this task, issue Card set C – *Measurements in standard form*. Learners should try to match these cards to the others on the table. There are six blank cards for learners to express their own estimates in standard form.

Next, ask each pair of learners to put aside the estimates that they produced themselves. They should then try to arrange the remaining cards in order of size. Thus learners should have the following items in order:

Nucleus of an atom	0.00000000000001	1×10^{-14} m
Length of a virus	0.0000002	2×10^{-7} m
Diameter of the eye of a fly	0.0008	8×10^{-4} m
Diameter of a 1p coin	0.02	2×10^{-2} m
Height of a door	2	2×10^{0} m
Height of a tall skyscraper	400	4×10^{2} m
Height of a mountain	8 000	8×10^{3} m
Distance between two furthest places on earth	20 000 000	2×10^{7} m
Distance from earth to moon	400 000 000	4×10^{8} m
Size of a galaxy	800 000 000 000 000 000 000	8×10^{20} m

Issue Card set D – *Comparisons*. Ask learners to place the arrow cards between each pair in the list to show how many times each item is greater in length than the item before. Two cards have been left blank for learners to complete.

Learners can check their answers using calculators. They may also enjoy making posters showing the completed arrangement of all the cards.

Learners who find the work straightforward may begin to manipulate numbers in standard form:

How many times taller is the mountain than the skyscraper?

How did you work this out?

How can you get this from heights expressed in standard form ($4 \times 10^2 \times ? = 8 \times 10^3$)?

Reviewing and extending learning

Discuss the various approaches that learners have used during the session and ask them to report back on what they have learned.

You may like to extend this work to explore the transformation of units. Use the fact that 1 km is 1 000 m.

How tall is the mountain in km?

How high is the skyscraper in km?

Can you give me that answer in standard form?

What learners might do next

Learners may enjoy making a poster displaying other quantities and numbers in standard form, for example comparing weights or capacities of various objects.

Further ideas

This activity uses multiple representations to deepen understanding of number. This type of activity may be used in any topic where a range of representations is used. Examples in this pack include:

N5 Understanding the laws of arithmetic;

A1 Interpreting algebraic expressions;

SS6 Representing 3D shapes.

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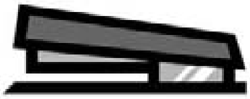
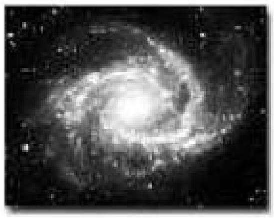
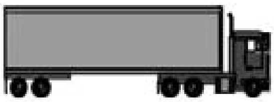
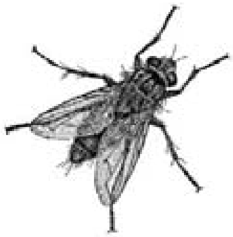
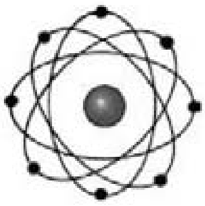
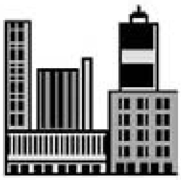
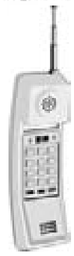
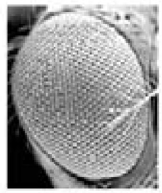
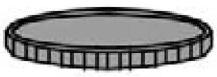
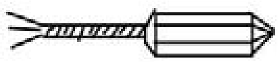
N4 Card set A – *Objects*

Wingspan of an aircraft 	Length of a stapler 	Height of a door 	Size of a galaxy 
Length of a truck 	Height of a mountain 	Length of a fly 	Nucleus of an atom 
Height of a tall skyscraper 	Height of a desk 	Length of a telephone 	Diameter of the eye of a fly 
Diameter of a 1p coin 	Distance from earth to moon 	Length of a virus 	Distance between two furthest places on earth 

N4 Card set B – *Measurements*

2 m	400 000 000 m
0.02 m	800 000 000 000 000 000 000 m
20 000 000 m	400 m
0.0008 m	0.0000000000001 m
8 000 m	0.0000002 m

N4 Card set C – *Measurements in standard form*

4×10^{8} m	2×10^{0} m
2×10^{-2} m	2×10^{-7} m
8×10^{3} m	4×10^{2} m
8×10^{20} m	8×10^{-4} m
2×10^{7} m	1×10^{-14} m

N4 Card set D – *Comparisons*

× 20 000 000	× 4 000
× 2 000 000 000 000	× 25
× 100	× 200
× 20 000 000	× 20
× 2500	× 20

N5 • Understanding the laws of arithmetic

Mathematical goals

To enable learners to:

- interpret numerical expressions using words and area representations;
- recognise the order of operations;
- recognise equivalent expressions;
- understand the distributive laws of multiplication and division over addition (the expansion of brackets).

Starting points

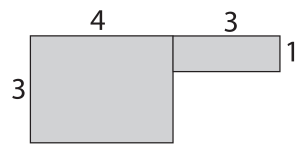
Most learners will have used the laws of arithmetic but some may have little understanding of the underlying principles. It is not enough to use a rote-learned rule like BODMAS or BIDMAS, as the introduction to the session will show.

The session assumes that learners are familiar with indices and with the area of simple compound shapes made up from rectangles joined together. However, you may need to check, through questioning, that learners understand these ideas:

What is the difference between 3×2 and 3^2?

Draw two different rectangles with an area of 36.

What is the area of this shape?



Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Calculations*;
- Card set B – *Areas*;
- Card set C – *Solutions*;
- the sheets for the brackets activity (optional).

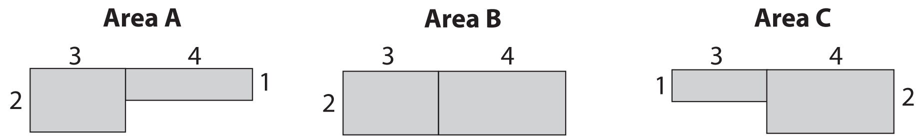
Time needed

Between half an hour and an hour.

Suggested approach

Beginning the session

Draw three compound shapes on the board.



Ask questions to probe learners' existing understanding.

If you work out $3 + 4 \times 2$, which area are you working out? Explain how you know.

If you work out $(3 + 4) \times 2$, which area are you working out? How do you know?

What answers does your calculator give for these questions?

Can you give me an expression for the other area?

What is the difference between $(2 + 3)^2$ and $2^2 + 3^2$? Can you show me a diagram to explain the difference?

If learners struggle with any of these questions, explain that you will leave it for now and return to it later in the session.

Ask learners if they have heard of BIDMAS (or BODMAS) and ask them to explain what it means. Explain the danger of using such a rule without understanding it. For example, write the following and ask the group to tell you where you have gone wrong.

$$
\begin{aligned}
3 \times \frac{(3+5)^2}{4} - 5 + 9 &= 3 \times \frac{8^2}{4} - 5 + 9 && \text{(B rackets)} \\
&= 3 \times \frac{64}{4} - 5 + 9 && \text{(I ndices – or 'powers')} \\
&= 3 \times 16 - 5 + 9 && \text{(D ivision)} \\
&= 48 - 5 + 9 && \text{(M ultiplication)} \\
&= 48 - 14 && \text{(A ddition)} \\
&= 34 && \text{(S ubtraction)}
\end{aligned}
$$

Working in groups

Arrange learners in pairs or groups of three. Give each group Card sets A (*Calculations*), B (*Areas*) and C (*Solutions*).

Ask learners to place the cards face up on the table and take it in turns to match them. If they feel that cards belong together, they should place them side-by-side, so that they are all visible. (Cards should not be stacked as this makes it impossible for you to monitor their work as you go round the room.) Each time that a learner matches two or three cards, they should try to explain to their partner(s) why the cards belong together. Encourage learners to challenge their partner(s) if they think an explanation is not clear enough.

Learners will soon realise that there are more *Calculations* cards than *Areas* and *Solutions* cards – do not comment at this stage. Learners will soon find that some areas may be obtained by more than one calculation and that they need to provide additional answers for themselves. The blank cards are provided for this.

Encourage learners to explain how they can immediately see when a *Calculations* card matches an *Areas* card, without working out answers. Ask them to look for alternative ways of finding the areas.

Learners who struggle with this activity could cut the compound shapes into rectangles and find the area of each rectangle before finding the area of the compound shape.

Learners who match the cards quickly may be challenged to move towards generalisation.

> What happens when we change the numbers?
>
> Suppose we change the 4 in every card to a 5? Will the calculations cards still match in the same way?
>
> Will this still be true when we change the 4 to a large number, a negative number or a decimal?
>
> Do the area pictures help to explain why this happens?

Reviewing and extending learning

When learners have completed their matching, return to the questions asked at the beginning. What answers can the learners now give?

Using mini-whiteboards and whole group questioning, begin to generalise the learning:

> Draw an area that requires this calculation: $3 \times (4 + 5)$. Write a different calculation that gives the same area.
>
> Draw an area that requires this calculation: $\frac{6 + 8}{2}$.
>
> Write a different calculation that gives the same area.
>
> Draw an area that requires this calculation: $(10 + 5)^2$. Write a different calculation that gives the same area.

Draw out the general learning points that have emerged:

The equivalence of multiplying by $\frac{1}{2}$ and dividing by 2.

The order of operations:

- brackets first;
- then powers or roots;
- then multiplication or division;
- then addition or subtraction.

Equivalent expressions:

$2 \times (3 + 4) = 2 \times 3 + 2 \times 4$
(multiplication is distributive over addition);

$\frac{3 + 4}{2} = \frac{3}{2} + \frac{4}{2}$
(division is distributive over addition);

$(3 + 4)^2 = 3^2 + 4^2 + 2 \times 3 \times 4$.

What learners might do next

Session **A1 Interpreting algebraic expressions** may be used to generalise what has been learned in this session.

The brackets activity

The brackets activity may be used to consolidate what has been learned.

Arrange learners into teams of three or four. Give each team a felt tip pen and two copies of Sheet A, four copies of Sheet B and four copies of Sheet C. These can be A5 size.

Sheet A	Sheet B	Sheet C
$2 + 3 \times 4 + 5$	$2 \times 3 + 4 \times 5$	$2 + 3 \times 4^2$

Call out a sheet name and a target number. Teams have to show how they can reach this target by adding brackets to the sheet.

For example, if you call out "Sheet A, 25", the first team to show you Sheet A with brackets in the following position will gain a point.

Sheet A
(2 + 3) x 4 + 5

Here are some answers, but learners may come up with others.

Sheet	Target	Method
A	29	$2 + 3 \times (4 + 5)$
A	45	$(2 + 3) \times (4 + 5)$
B	26	$(2 \times 3) + (4 \times 5)$ or no brackets
B	46	$2 \times (3 + 4 \times 5)$
B	50	$(2 \times 3 + 4) \times 5$
B	70	$2 \times (3 + 4) \times 5$
C	80	$(2 + 3) \times 4^2$
C	146	$2 + (3 \times 4)^2$
C	196	$(2 + 3 \times 4)^2$
C	400	$((2 + 3) \times 4)^2$

Further ideas

This activity uses multiple representations to deepen understanding of number operations. This type of activity may be used in any topic where a range of representations is used. Examples in this pack include:

A1 Interpreting algebraic expressions;

SS6 Representing 3D shapes;

S4 Understanding mean, median, mode and range.

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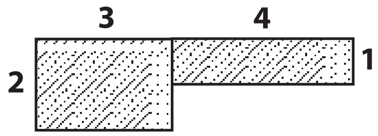
N5 Card set A – *Calculations*

A1 $3^2 + 4^2$	**A2** $2 \times (3 + 4)$
A3 $(3 + 4)^2$	**A4** 3×4^2
A5 $(3 \times 4)^2$	**A6** $\frac{3}{2} + \frac{4}{2}$
A7 $2 \times 3 + 4$	**A8** $4 + 3 \times 2$
A9 $3^2 \times 4^2$	**A10** $2 \times 3 + 2 \times 4$
A11 $\frac{1}{2}(3 + 4)$	**A12** $3^2 + 4^2 + 2 \times 3 \times 4$
A13 $\frac{3 + 4}{2}$	**A14** $\frac{3}{2} + 4$

N5 Card set B – *Areas*

B1



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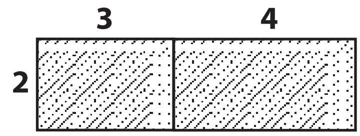

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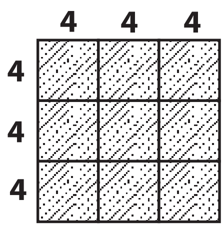

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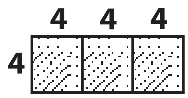

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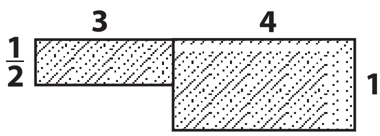

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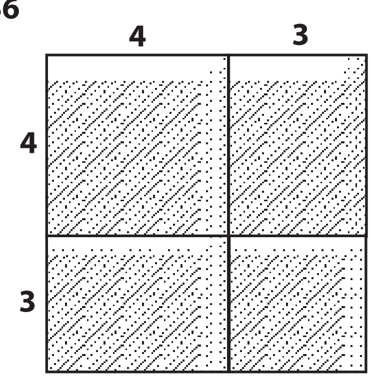
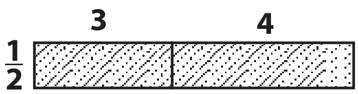
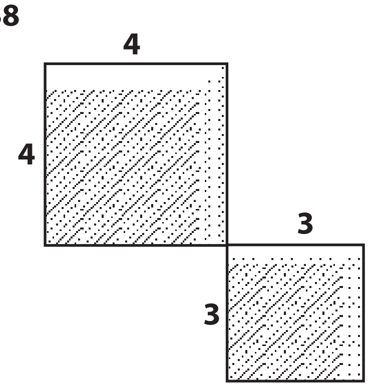


B7



B8



N5 Card set C – *Solutions*

144	48
49	3.5
5.5	14

N6 • Developing proportional reasoning

Mathematical goals

To help learners to:

- reflect on the reasoning they currently use when solving proportion problems;
- examine proportion problems and appreciate their multiplicative structure;
- create their own variants of proportion problems.

Starting points

Proportional reasoning is notoriously difficult for many learners. Many have difficulty in recognising the multiplicative structures that underlie proportion problems. Instead, they use addition methods, or informal methods using doubling, halving and adding. This session aims to expose and build on this prior learning.

Learners are given four direct proportion problems to solve, taken from different areas of the mathematics curriculum. They then compare their methods for solving these with methods produced by other learners. This leads to a discussion that compares the use of more primitive informal methods that use adding, doubling and halving with the use of more sophisticated methods that use multiplication.

Materials required

- OHT 1 – *Solving proportion problems in one step*;
- OHT 2 – *Solving proportion problems in two steps*;
- an OHT of 'Paint prices' from Sheet 1 – *Problems* (page 1).

For each learner you will need:

- Sheet 1 – *Problems* (pages 1 and 2);
- Sheet 2 – *Sample work*;
- Sheet 3 – *Making up your own questions*;
- a calculator.

Time needed

At least 1 hour. If time is short, fewer problems may be tackled.

Suggested approach

Beginning the session

Give out copies of Sheet 1 – *Problems* (pages 1 and 2). Allow learners time to tackle the problems individually or in pairs. They may use calculators if they wish. Do not comment on learners' solution strategies at this stage as the purpose of the session is to compare different strategies.

Working in groups

Hand out copies of Sheet 2 – *Sample work*. Invite learners to assess these pieces of work.

They should attempt to:

- correct the work;
- write advice to the learner, identifying and explaining the errors that have been made and how the solution strategies can be improved.

The pieces of work illustrate the following issues:

Recipe

The learner has answered part 1 correctly, but part 2 incorrectly. For both parts, the learner has adopted an addition strategy. In part 1, she has reasoned that, because 10 = 4 + 4 + half of 4, then the quantities in the recipe may be deduced by doubling, then adding one half as much again. This is a correct strategy and is one that is helpful when working out calculations mentally. When a calculator is available, however, it is not the most efficient method.

In the second problem, the learner has followed the same strategy but has then added 'tops and bottoms'.

Paint prices

In part 1 the learner has chosen, wrongly, to divide rather than to multiply. This may be because the learner believes that 'division will make numbers smaller' and so 15 ÷ 0.6 will be less than 15. (Multiplication has been rejected on the grounds that it will result in a number greater than 15.)

In parts 2 and 3, a 'halving and adding' strategy has been correctly used.

Part 4 is correct; a multiplying strategy has been correctly used.

Enlarging a photograph to make a poster

In these two commonly-found answers, an additive strategy has been incorrectly used.

Advertising

The percentages have been correctly calculated using an addition strategy. With a calculator, a more efficient strategy would have been to simply calculate the missing entries by multiplying or dividing by 3.6

Whole group discussion

Place an OHT of the 'Paint prices' problem from Sheet 1 – *Problems* onto the overhead projector and ask learners to share their comments about the sample work. Invite learners to suggest better ways of doing the problems, assuming that a calculator is available. In particular, challenge them to find a simple and efficient way of obtaining each answer using a calculator. For example, the missing paint prices can be found quite easily by multiplying every capacity by 15.

Ask learners if they noticed that the problems all have something in common.

Explain the following ideas, asking learners to contribute to each step. Write their responses on the board.

All the problems we have looked at in this session have involved two quantities. Can you help me list them?

Number of pancakes	Amount of ingredients
Quantity of paint	Cost of paint
Height of poster	Width of poster
Percentage of money spent	Angle in the pie chart

These are proportional situations. If we double the first quantity, we double the second. If we plotted a graph of the two quantities, we would get a straight line through the origin.

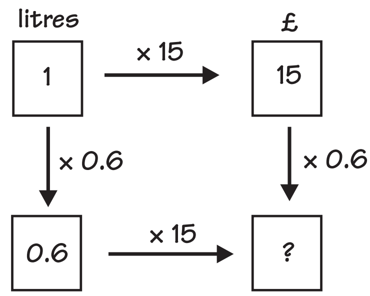
Can you suggest some more pairs of quantities that are proportional?

Can you suggest some that are not?

Learners may suggest other proportional pairs such as distance travelled and time taken (assuming constant speed), and others that are not, such as speed of travel and time taken (this is inversely proportional).

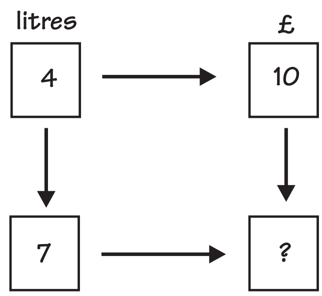
Show OHT 1 – *Solving proportion problems in one step* (or draw the diagram on the board). Explain that you will show a method for working out proportional problems using this diagram.

> Look at the paint problem. The two quantities are amount bought (litres) and cost (£).
>
> 1 litre costs £15. (Fill in the top two boxes in the diagram).
> We need to know how much 0.6 litres costs. (Fill in the remaining quantities in the diagram, explaining the reasoning.)



Explain how proportion problems are really just multiplication problems and show how the multipliers may be found and used. The vertical multipliers are dimensionless. The horizontal ones are rates (here £ per litre).

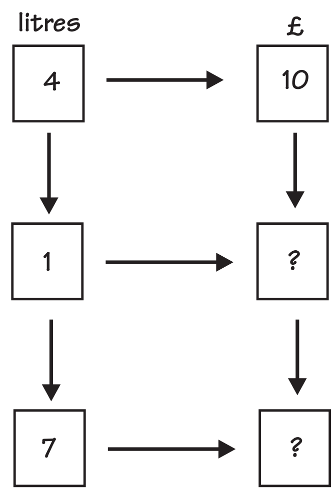
Create a new problem by changing the numbers, for example as shown here. Ask learners to state the new problem in their own words and to suggest the multipliers which, this time, are not so obvious. Some may see that the rows involve multiplying by $\frac{10}{4}$ or 2.5, and the columns by $\frac{7}{4}$ or 1.75.



In the same way, create additional problems using different sets of numbers, some of which use multipliers that are less than 1.

Learners who struggle may like to use two steps (using OHT 2 – *Solving proportion problems in two steps*), first dividing by 4 to find the price of 1 litre, then multiplying by 7 to find the price of 7 litres, as shown here.

Ask learners to return to the original problems and, using this diagram method, to solve any they could not do earlier.



Reviewing and extending learning

Ask learners to tell you how they can recognise when a situation is a direct proportion and when it is not.

When one value is zero, so is the other.

When one variable doubles, so does the other.

The graph of one variable against the other is a straight line through the origin.

The formula is $y = ax$.

What learners might do next

Issue learners with Sheet 3 – *Making up your own questions*. This sheet contains a selection of problems with numbers missing. Learners need to:

- decide which situations are direct proportions and which are not;
- write their own numbers in the spaces;
- solve the problems that are created, using both their own informal methods and the diagram method.

Encourage learners to use at least two sets of numbers for each problem. One set should make the problem quite easy (but not trivial), and one set should make it quite difficult.

If some learners are more able, ask them to write variable names for each space (e.g. x, y and z) and to write solutions for each situation using these letters.

Further ideas

This session involves analysing and comparing different methods for solving problems. This is a powerful idea that may be used in any topic. Why not give your learners a completed examination paper to mark and comment on? This is very useful if the answers that you provide reveal common misconceptions.

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N6 Sheet 1 – *Problems* (page 1)

RECIPE

Pancake mix

For 4 pancakes you will need:

6 dessertspoons of flour

$\frac{1}{4}$ litre milk

1 pinch salt

1 egg

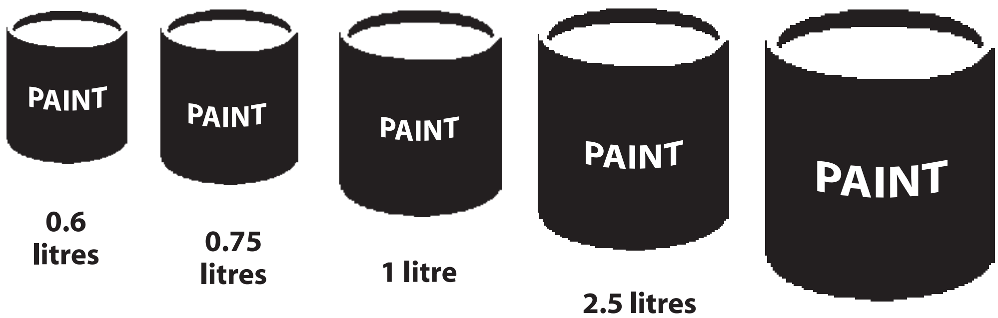
You want to make 10 pancakes.

1. How much flour do you need?
2. How much milk do you need?

PAINT PRICES

Calculate the missing prices of the paint cans below.
The prices are proportional to the amount of paint in the can.

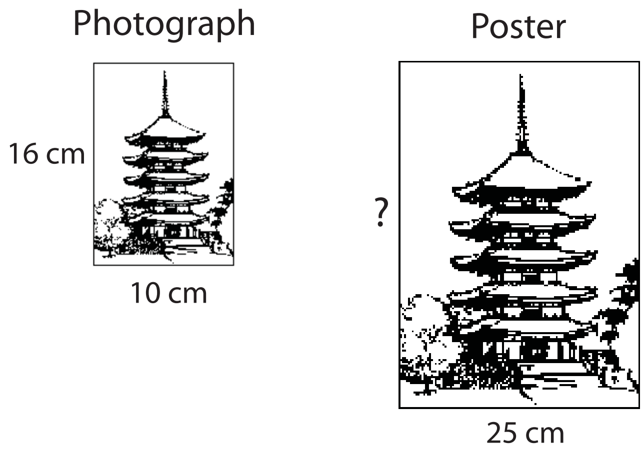
£ …	**£ …**	**£15**	**£ …**	**£ …**



0.6 litres	**0.75 litres**	**1 litre**	**2.5 litres**	**4.54 litres (1 gallon)**

N6 Sheet 1 – *Problems* (page 2)

ENLARGING A PHOTOGRAPH TO MAKE A POSTER

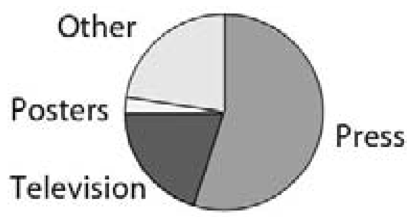


1. The photograph is enlarged to make a poster.
   The photograph is 10 cm wide and 16 cm high.
   The poster is 25 cm wide.
   How high is the poster?

2. In the poster, the building is 30 cm tall.
   How tall is it in the photograph?

ADVERTISING

	% spent	Angle in pie chart
Press	55	
Television	20	72°
Posters		9°
Other		



The pie chart shows the proportion spent on advertising in different media in one year.

Calculate the missing entries in the table.

N6 Sheet 2 – *Sample work*

(i) Mark the answers right or wrong.

(ii) Find the causes of the mistakes.

(iii) Write down your advice to the learner, explaining how the work should be improved, even when the answer is right.

Recipe

① 10 pancakes = 6 + 6 + 3 = 15 spoons of flour

② = $\frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{3}{16}$ pints of milk.

Paint prices

1 0.6 litres 15 ÷ 0.6 = 25p

2 0.75 litres = $\frac{1}{2} + \frac{1}{4}$

= £7.50 + £3.75 = £11.25

3. 2.5 litres = $2 + \frac{1}{2}$

= £30 + £7.50 = £37.50

4 4.54 litres = 15 × 4.54

= 68.1

Enlarging a photograph to make a poster

The poster is 15 cm bigger

(1) 16 + 15 = 31 cm high.

(2) 30 − 15 = 15 cm high.

Advertising

20% ⇒ 72°
10% ⇒ 36°
5% ⇒ 18°

Press = 72° + 72° + 36° + 18° ⇒ 198°

Poster = 9° ⇒ 2.5%

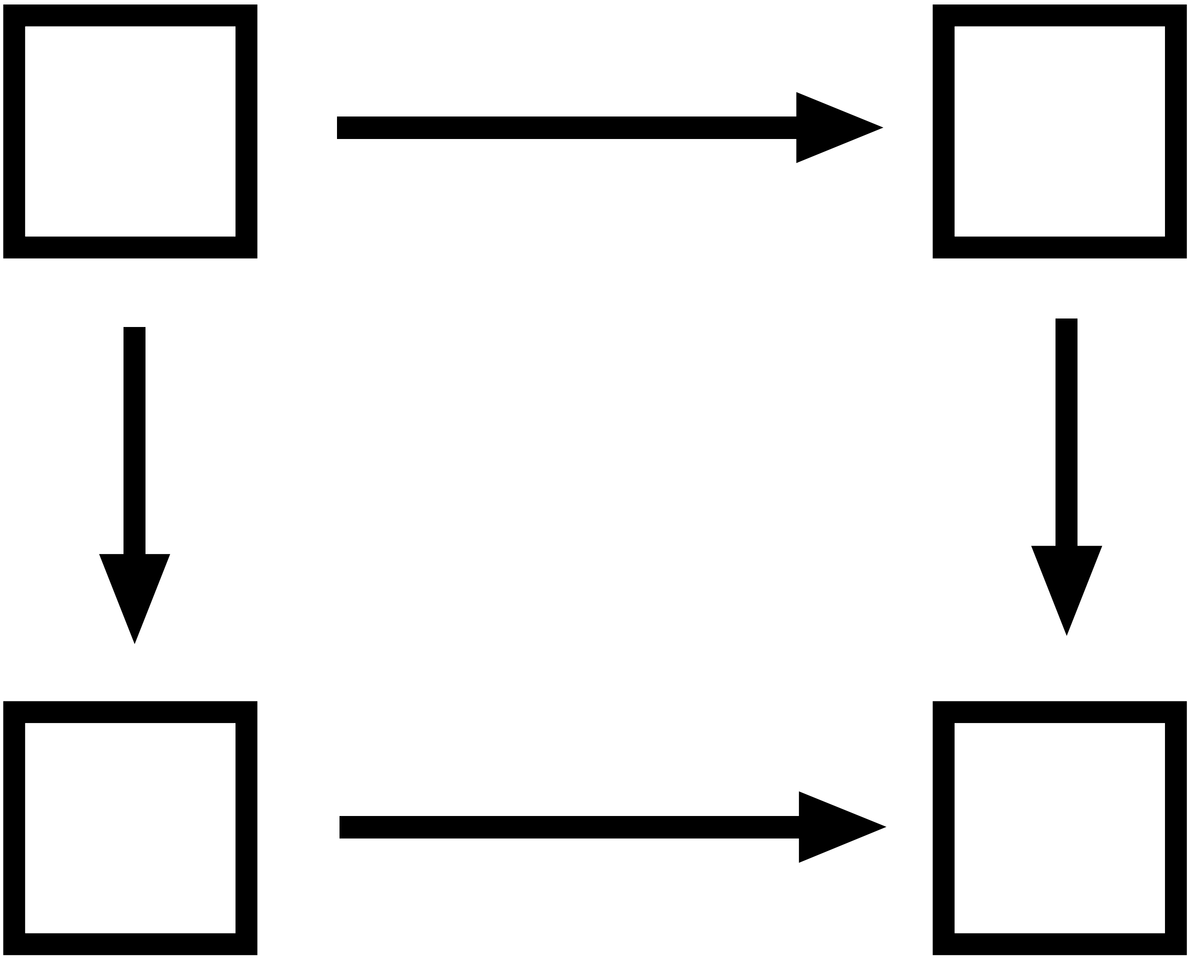
Other = 100 − 77.5 = 22.5%

= 20% + 2.5%

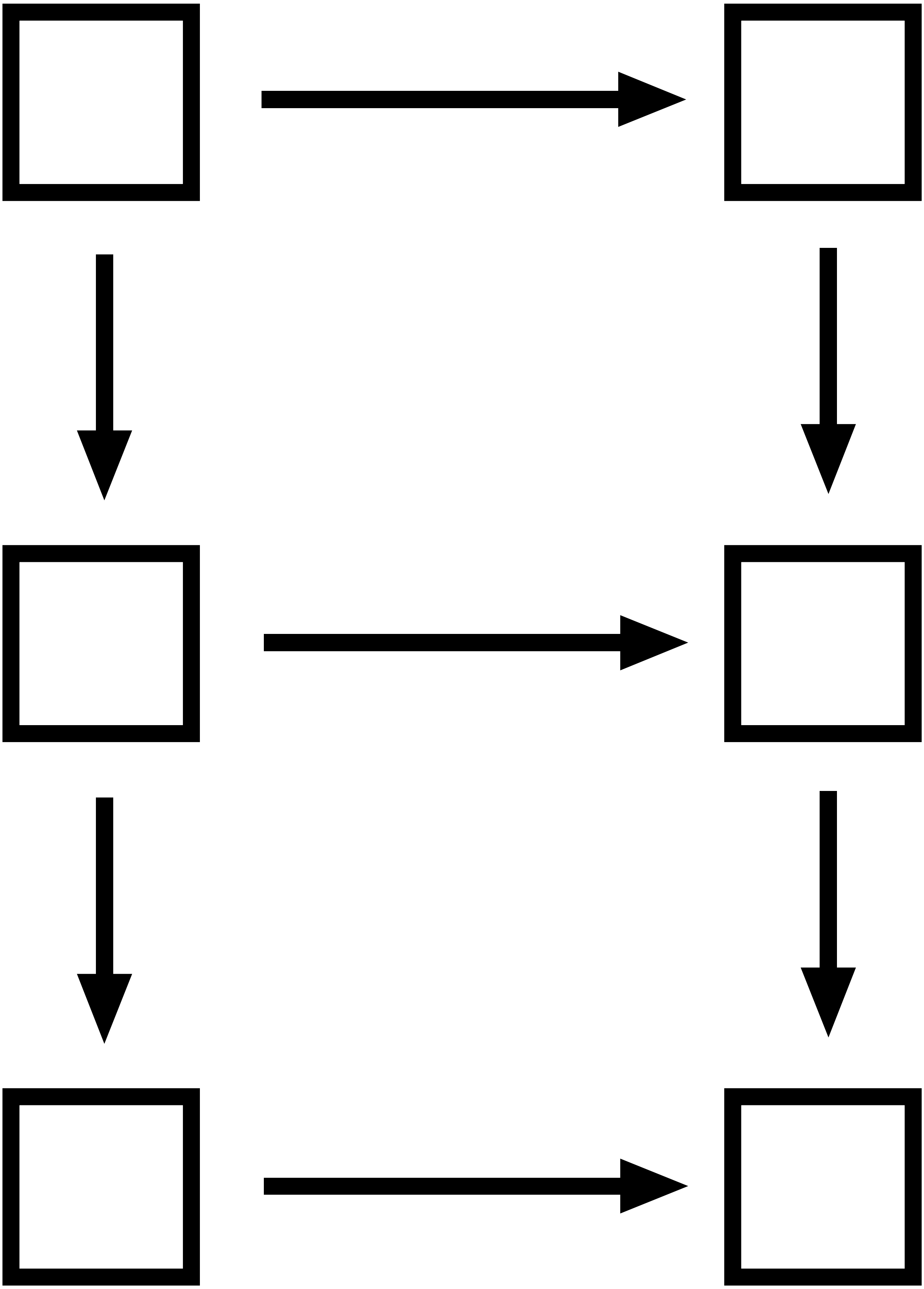
⇒ 72° + 9° = 81°

55 +
20
2.5
77.5

N6 OHT 1 – *Solving proportion problems in one step*



N6 OHT 2 – *Solving proportion problems in two steps*



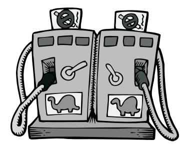
N6 Sheet 3 – *Making up your own questions*

CYCLE

It takes .... minutes to cycle ...... miles.

At the same speed, how long does it take to cycle ...... miles?

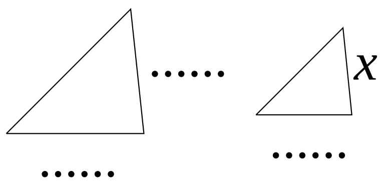
PETROL



..... litres cost £ .....

How much will ...... litres cost?

TRIANGLES



These two triangles are similar.

Calculate the length marked x.

DRIVING

If I drive at ...... miles per hour, the journey will take .... hours.

How long will it take if I drive at .... miles per hour?

LINE

A straight line passes through the points (0,0) and (...... , ...).

It also passes through the point (...... , y).

Calculate the value of y.

FIRE

It would take ..... minutes to vacate a building if we put in ..... fire escapes.

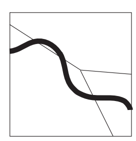
How long would it take with ..... fire escapes?

MONEY

£ ..... is worth the same as ..... dollars.

If I change £ ....., how many dollars will I get?

MAP



A road ..... cm long on a map is ..... km long in real life.

A river is ..... cm long on the map. How long is the real river?

N7 • Using percentages to increase quantities

Mathematical goals

To enable learners to:

- make links between percentages, decimals and fractions;
- represent percentage increase and decrease as multiplication;
- recognise the inverse relationship between increases and decreases.

Starting points

Most learners will have met these concepts before. Many, however, will have been introduced to the ideas in a procedural manner ("this is how you calculate a percentage of a quantity") rather than in a conceptual one. Gaps and misconceptions will remain. Typically, for example, we find that learners believe an increase of 50% followed by a decrease in 50% takes us back to the original value. This session will confront such misconceptions and build new conceptual links between percentages, decimals and fractions.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Money*;
- Card set B – *Percentages*;
- Card set C – *Words*;
- Card set D – *Decimals*;
- Card set E – *Fractions*;
- calculators.

Time needed

From 1 to 2 hours, depending on how many card sets are used.

Suggested approach

Beginning the session

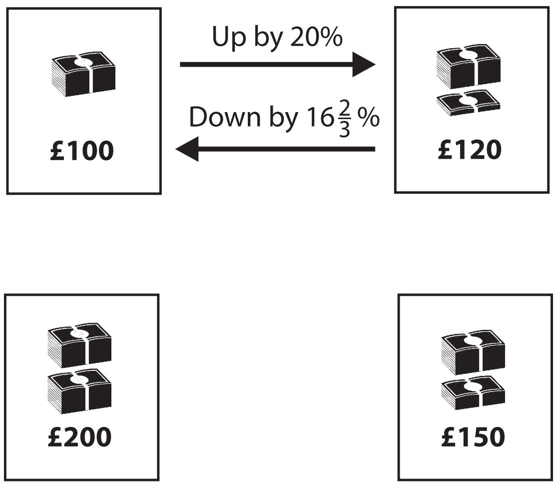
Ask learners to tackle the following question on their own, without discussion.

> In a sale, the prices in a shop were all reduced by 33%.
> After the sale they were all increased by 50%.
> What was the overall effect on the shop prices?
> Explain how you know.

Experience suggests that most learners will answer that prices have risen by 17%. The intention here is simply to expose existing thinking. Towards the end of the session, learners will reconsider their answers and put them right.

Working in groups

Ask learners to work in pairs or threes. They will need plenty of table space for the activity, so you may need to push tables together. Give each group Card sets A – *Money* and B – *Percentages*. Ask learners to place the money cards in the form of a square on the table, so that their values increase in a clockwise direction, starting at top left, as shown in the diagram.



Learners must take it in turns to place pairs of *Percentages* cards between each pair of *Money* cards to show the correct percentage increase or decrease. Pairs may be horizontal, vertical or diagonal. Blank cards are provided if learners wish to make new cards.

If you think learners will struggle with this, you may like to start with just three *Money* cards (£100, £150, £200) and the corresponding *Words* (Card set C), *Decimals* (Card set D) and/or *Fractions* (Card set E) cards. If you think that some learners may find the work straightforward, you may like to make the *Money* cards more complex (e.g. 80p, 96p, £1.60, £1.20).

Typically, learners make the mistake of pairing an increase of 50%

with a decrease of 50%, and so on. It is important that you do not comment on this at this stage. You should wait to see if learners can correct these mistakes for themselves, as more cards are added.

When learners have done this, introduce Card sets C and D. Ask learners to add these to the arrangement on the table. These cards provide learners with different ways of interpreting the situation. Allow access to calculators to facilitate the arithmetic.

Finally, when learners have completed this stage, give out Card set E and ask learners to place these in position.

As you monitor the work of the groups, listen to their discussions. Help learners to look for patterns and generalisations in their results, perhaps by highlighting some yourself. Learners may notice the following.

- An increase of, say, 33% is equivalent to multiplying by 1.33. So, an easy way to increase by n% is to write n% as a decimal, add one and multiply by this number.
- A decrease of, say, 33% is equivalent to multiplying by (1–0.33) = 0.67. So, an easy way of decreasing by n% is to write n% as a decimal, subtract it from one and multiply by the resulting number.
- The inverse of an increase by a percentage is not a decrease by the same percentage.
- There is a clear pattern in the pairs of words that represent inverse functions:

Doubled	Down by one half
Up by one half	Down by one third
Up by one third	Down by one quarter
Up by one quarter	Down by one fifth
Up by one fifth	Down by one sixth

- This pattern is less clear when we consider decimals, although a calculator enables us to easily show that each pair multiplies to give 1.

$\times 2$	$\times 0.5$	and $2 \times 0.5 = 1$
$\times 1.5$	$\times 0.\dot{6}$	and $1.5 \times 0.\dot{6} = 1$
$\times 1.\dot{3}$	$\times 0.75$	and $1.\dot{3} \times 0.75 = 1$
$\times 1.25$	$\times 0.8$	and $1.25 \times 0.8 = 1$
$\times 1.2$	$\times 0.8\dot{3}$	and $1.2 \times 0.8\dot{3} = 1$

- The pattern in the pairs of fraction multipliers is very easy to see:

$\times \frac{2}{1}$	$\times \frac{1}{2}$
$\times \frac{3}{2}$	$\times \frac{2}{3}$

$\times \frac{4}{3}$ $\times \frac{3}{4}$

. . . and so on.

Reviewing and extending learning

Discuss and generalise what has been learned with the whole group, using mini-whiteboards and questions pitched at appropriate levels.

If a price increases by 10% . . .

- How can you write that in words?
- How can you write that as a decimal multiplication?
- How can you write that as a fraction multiplication?

How much will the price need to go down to get back to the original price?

- How can you write that in words?
- How can you write that as a decimal multiplication?
- How can you write that as a fraction multiplication?

Finally, reconsider the problem that began this session. How would learners now answer this differently?

What learners might do next

If you want to continue the session on another day, then begin with a different, harder, set of *Money* cards. Alternatively, you could use shapes and discuss enlargement by different scale factors.

You could also ask learners to create their own sets of cards using the generalisations they have deduced in this session.

Further ideas

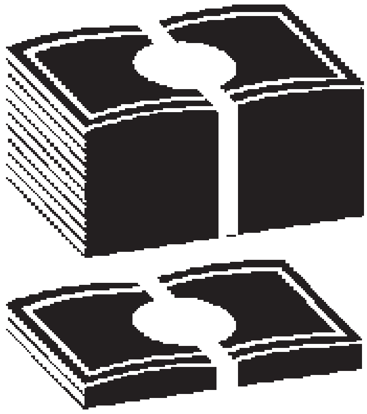
This session uses multiple representations of states and transformations. Similar activities in other mathematical contexts are included in this pack. For example:

N8 Using directed numbers in context
(states are temperatures; transformations are rises and falls);

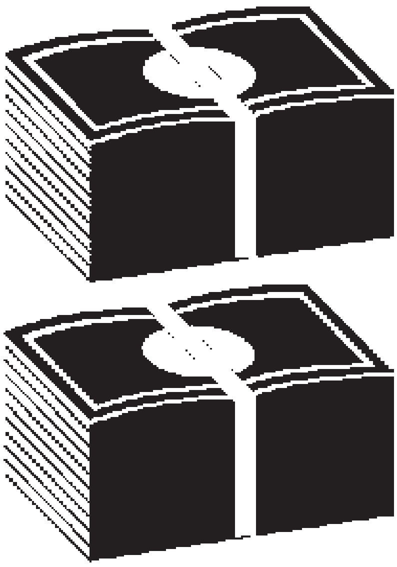
SS7 Transforming shapes
(states are shapes; transformations are translations, reflections and rotations).

N7 Card set A – *Money*

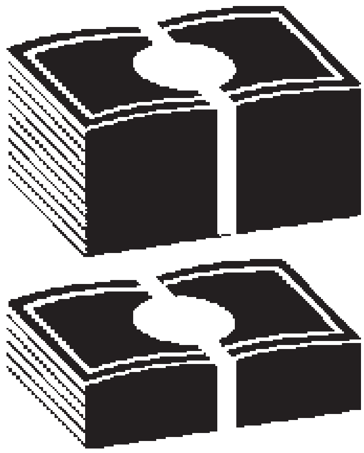
£100



£120



£200



£150

N7 Card set B – *Percentages*

Down by 50%	Down by 20%
Up by 25%	Up by 20%
Down by $33\frac{1}{3}$%	Down by $16\frac{2}{3}$%
Down by 25%	Up by 50%
Up by $33\frac{1}{3}$%	Up by 100%

N7 Card set C – *Words*

Up by one half	Down by one sixth
Down by one third	Doubled
Up by one fifth	Up by one quarter
Down by one fifth	Down by one quarter
Down by one half	Up by one third

N7 Card set D – *Decimals*

$\times 1.2$	$\times 0.\dot{6}$
$\times 0.75$	$\times 2$
$\times 1.5$	$\times 0.8\dot{3}$
$\times 0.8$	$\times 1.\dot{3}$
$\times 0.5$	$\times 1.25$

N7 Card set E – *Fractions*

$\times \frac{2}{1}$	$\times \frac{3}{2}$
$\times \frac{4}{5}$	$\times \frac{4}{3}$
$\times \frac{2}{3}$	$\times \frac{5}{6}$
$\times \frac{5}{4}$	$\times \frac{3}{4}$
$\times \frac{6}{5}$	$\times \frac{1}{2}$

N8 • Using directed numbers in context

Mathematical goals

To help learners to:

- understand and use directed numbers in the context of temperatures.

Starting points

This session requires no prior knowledge.

Learners are provided with the temperatures in four cities. Temperature differences between these and four other cities are also given. Learners are asked to find missing temperatures and temperature differences. This leads them to consider one interpretation of addition and subtraction with negative numbers.

Materials required

For each small group of learners you will need:

- Card set A – *City temperatures*;
- Card set B – *Temperature changes*;

and possibly:

- a temperature scale cut from Sheet 1 – *Temperature scales*.

Time needed

Approximately 1 hour.

Suggested approach

Beginning the session

Ask four learners to come to the front of the room and give them each a card on which you have written the name of a city and a temperature.

For example, you could use the following:

London	13 °C
Moscow	–8 °C
Montreal	–11 °C
Madrid	20 °C

Ask the learners to stand in order of temperature – coldest to hottest.

Ask the rest of the group questions such as the following:

If I travel instantly from London to Madrid:

- Does the temperature rise or fall? By how much?
- Madrid to Moscow?
- Moscow to Montreal?

Cairo is 30° warmer than Montreal.

- What is the temperature in Cairo?

After each question ask learners to explain how they worked it out.

You may find it helpful to use an OHT of one of the temperature scales to help with the explanation.

Working in groups

Ask learners to sit in pairs or threes. Give out Card set A – *City temperatures* and Card set B – *Temperature changes*. Card set B shows temperature changes as you travel from one city to another.

Ask learners to link the *City temperatures* cards to the *Temperature changes* cards. As they position each card, they should be able to work out either an unknown city temperature or an unknown temperature change and write it in the space on the card.

Ask learners to take it in turns to place the *Temperature changes* cards in their correct positions. As they place a card, they should explain to their partner(s) why they have placed the card in that position. When they have given their explanation, their partner(s) should either challenge what they have said or say why they agree.

While they work on the task, encourage learners to check their work using different routes with the cards.

If learners have difficulty, give them a temperature scale from Sheet 1 to help them illustrate the changes.

Reviewing and extending learning

Explain to the whole group that there are several ways of calculating missing numbers. Ask learners to describe the ways they have used for some specific examples. Elicit more than one method for each case and use these to develop the structure of the situations as directed number calculations. For example:

Final temperature missing

Verkhoyansk	Verkhoyansk to Wellington	Wellington
–40 °C	+55 °C	?

Starting temperature + change = final temperature:
–40 + (+55) = +15

Starting temperature missing

Khartoum	Khartoum to Sydney	Sydney
?	–10 °C	+25 °C

Final temperature – change = starting temperature:
+25 – (–10) = +35

Change missing

Sydney	Sydney to Verkhoyansk	Verkhoyansk
25 °C	?	–40 °C

Final temperature – initial temperature = change in temperature:
–40 – (+25) = – 65

Combining changes

Khartoum to Sydney	Sydney to Moscow	Khartoum to Moscow
–10 °C	–35 °C	?

First change + second change = combined change:
–10 + (–35) = –45

Ask learners to generalise rules for adding or subtracting directed quantities from these or similar examples. For instance, the four examples shown above illustrate the effect of adding a positive, subtracting a negative, subtracting a positive, and adding a negative.

What learners might do next

Remind learners of other situations and contexts where negative numbers appear e.g. time differences, bank credits and debits etc. For given calculations, can they think of suitable problems?

For example: – 25 + (– 30) = – 55

I have two bank accounts. One has an overdraft of £25. The other has an overdraft of £30. Altogether I have an overdraft of £55.

Session **N9 Evaluating directed number statements** would be a useful follow-up to this session.

There are several websites that give information on time differences between Greenwich Mean Time and other world cities. Learners could investigate these and create problems involving time differences.

Further ideas

This session uses representations of states (temperatures) and transformations (temperature changes). Similar activities in other mathematical contexts are included in this pack. For example:

SS7 Transforming shapes
(states are shapes; transformations are translations, reflections rotations and enlargements);

N7 Using percentages to increase quantities
(states are money values; transformations are percentage increases/decreases).

N8 Card set A – *City temperatures*

Anchorage	**London** –5 °C
Moscow	**Verkhoyansk** –40 °C
Rio 30 °C	**Khartoum** ....
Sydney +25 °C	**Wellington** ....

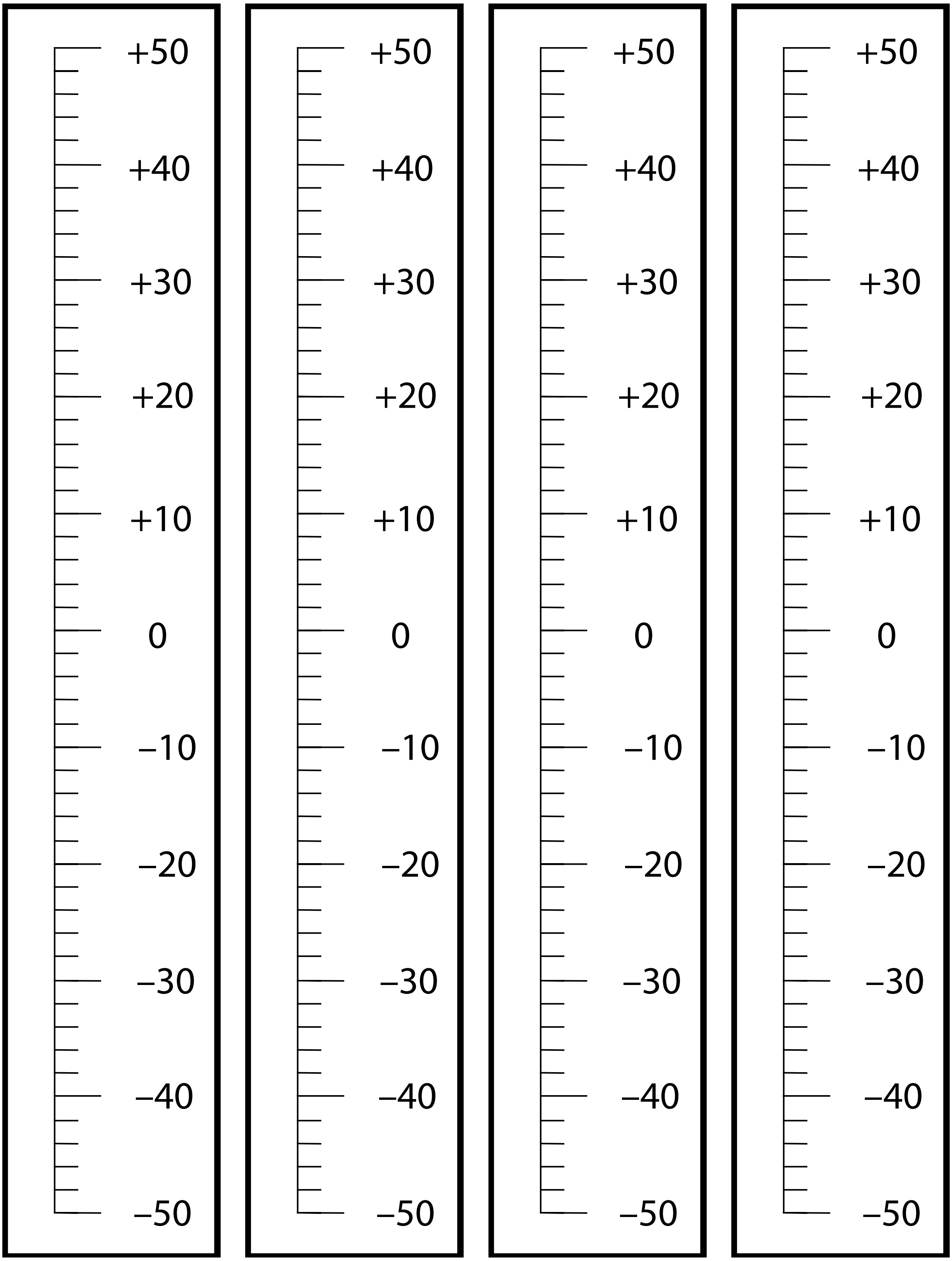
N8 Card set B – *Temperature changes*

Anchorage to London +15 °C	Wellington to Sydney +10 °C
London to Moscow .... °C	Anchorage to Rio +50 °C
Moscow to Verkhoyansk –30 °C	London to Khartoum +40 °C
Rio to Khartoum .... °C	Moscow to Sydney +35 °C
Khartoum to Sydney –10 °C	Verkhoyansk to Wellington +55 °C

N8 Card set B – *Temperature changes* (continued)

Anchorage to Khartoum .... °C	Rio to London .... °C
London to Sydney .... °C	Khartoum to Moscow .... °C
Moscow to Wellington .... °C	Sydney to Verkhoyansk .... °C

N8 Sheet 1 – *Temperature scales* (four copies)



N9 • Evaluating directed number statements

Mathematical goals

To enable learners to:

- make valid generalisations about the effect of operations on directed numbers.

Starting points

It is helpful if learners have already attempted to use directed quantities in contexts (e.g. money or temperature) before attempting this session. A suitable session to prepare for this is **N8 Using directed numbers in context**.

In this session, learners are provided with a collection of statements. They have to decide whether these statements are always, sometimes or never true and justify their choices with examples and counter-examples.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Statements*;
- large sheet of paper for making a poster;
- felt tip pens;
- glue stick.

and optionally:

- Sheet 1 – *Addition grid; subtraction grid; multiplication grid; division grid*;
- calculator.

Time needed

Approximately 1 hour.

Suggested approach

Beginning the session

Using mini-whiteboards, ask learners to show you simple addition, subtraction, multiplication and division questions that give the answers 12, 0, and –12.

> e.g. Give me an addition question where the answer is –12. Can you find a harder example?

Working in groups

Ask learners to work in pairs. Give each pair Card set A – *Statements*, a large sheet of paper, glue stick and felt tip pens.

If learners are likely to struggle, then just give the eight cards on addition and subtraction to start with.

Ask learners to divide the poster into three sections headed 'Always true', 'Sometimes true' or 'Never true'.

The object of the activity is for each pair of learners to produce a poster which shows each statement classified according to whether it is always, sometimes or never true and furthermore:

- if it is sometimes true, to write examples around the statement to show when it is true and when it is not true;
- if it is always true, to give a variety of examples demonstrating that it is true, using large numbers and decimals, if possible;
- if it is never true, to say how we can be sure that this is the case.

If learners have difficulties or make many mistakes, give them Sheet 1 – *Addition grid; subtraction grid; multiplication grid; division grid*. They can complete the grids quickly, using the patterns that rapidly become apparent. There are a few blank cells in each grid for learners to complete themselves. The results from these grids will provide learners with further examples they can use to check their posters.

Learners may also like to find ways of checking with a calculator. This requires careful use of the (+/–) key.

Reviewing and extending learning

Working with the whole group, write on the board a list of statements that learners think are always true and ask for examples to justify this. There are some interesting additional discussions that might arise, such as the meaning of '0 ÷ 0'.

What learners might do next

Learners may be asked to create money or temperature problems that lead to the calculations on the cards.

Further ideas

This activity is about examining a mathematical statement and deciding on its truth or falsehood. This idea may be used in many other topics and levels. Examples in this pack include:

A4 **Evaluating algebraic expressions;**

SS4 **Evaluating statements about length and area;**

S2 **Evaluating probability statements.**

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N9 Card set A – *Statements*

A **(–5) + (–6)** **If you add two negative numbers you get a negative answer.**	**B** **(–5) + (+7)** **If you add a negative number and a positive number you get a positive answer.**
C **(–5) – (+4)** **If you subtract a positive number from a negative number you get a negative answer.**	**D** **(–5) – (–8)** **If you subtract a negative number from a negative number you get a positive answer.**
E **(+10) – (+5)** **If you subtract a positive number from a positive number you get a positive answer.**	**F** **(+8) – (–6)** **If you subtract a negative number from a positive number you get a positive answer.**
G **5 + (–8) = 5 – (+8)** **Adding a negative is like subtracting a positive.**	**H** **5 – (–8) = 5 + 8** **Subtracting a negative is like adding a positive.**

N9 Card set A – *Statements* (continued)

I **(−10) × (−5)** **If you multiply two negative numbers you get a negative answer.**	**J** **(−10) × (+6)** **If you multiply a negative number and a positive number you get a positive answer.**
K **(+12) ÷ (−4)** **If you divide a positive number by a negative number you get a negative answer.**	**L** **(−12) ÷ (+4)** **If you divide a negative number by a positive number you get a negative answer.**
M **(+12) ÷ (+4)** **If you divide a positive number by a positive number you get a negative answer.**	**N** **(−12) ÷ (−4)** **If you divide a negative number by a negative number you get a negative answer.**

N9 Sheet 1

Addition grid

3 + 3 =	3 + 2 =	3 + 1 =	3 + 0 =	3 + (–1) =	3 + (–2) =	3 + (–3) =
2 + 3 =	2 + 2 =	2 + 1 =	2 + 0 =	2 + (–1) =	2 + (–2) =	2 + (–3) =
1 + 3 =	1 + 2 =	1 + 1 =	1 + 0 =	1 + (–1) =	1 + (–2) =	1 + (–3) =
0 + 3 =	0 + 2 =	0 + 1 =	0 + 0 =	0 + (–1) =	0 + (–2) =	0 + (–3) =
–1 + 3 =	–1 + 2 =	–1 + 1 =	–1 + 0 =	–1 + (–1) =	–1 + (–2) =	–1 + (–3) =
–2 + 3 =	–2 + 2 =	–2 + 1 =	–2 + 0 =	–2 + (–1) =		
–3 + 3 =	–3 + 2 =	–3 + 1 =	–3 + 0 =	–3 + (–1) =		–3 + (–3) =

Subtraction grid

3 – 3 =	3 – 2 =	3 – 1 =	3 – 0 =	3 – (–1) =	3 – (–2) =	3 – (–3) =
2 – 3 =	2 – 2 =	2 – 1 =	2 – 0 =	2 – (–1) =	2 – (–2) =	2 – (–3) =
1 – 3 =	1 – 2 =	1 – 1 =	1 – 0 =	1 – (–1) =	1 – (–2) =	1 – (–3) =
0 – 3 =	0 – 2 =	0 – 1 =	0 – 0 =	0 – (–1) =	0 – (–2) =	0 – (–3) =
–1 – 3 =	–1 – 2 =	–1 – 1 =	–1 – 0 =	–1 – (–1) =	–1 – (–2) =	–1 – (–3) =
–2 – 3 =	–2 – 2 =	–2 – 1 =	–2 – 0 =	–2 – (–1) =		
–3 – 3 =	–3 – 2 =	–3 – 1 =	–3 – 0 =	–3 – (–1) =		–3 – (–3) =

N9 Sheet 1 (continued)

Multiplication grid

3 × 3 =	3 × 2 =	3 × 1 =	3 × 0 =	3 × (–1) =	3 × (–2) =	3 × (–3) =
2 × 3 =	2 × 2 =	2 × 1 =	2 × 0 =	2 × (–1) =	2 × (–2) =	2 × (–3) =
1 × 3 =	1 × 2 =	1 × 1 =	1 × 0 =	1 × (–1) =	1 × (–2) =	1 × (–3) =
0 × 3 =	0 × 2 =	0 × 1 =	0 × 0 =	0 × (–1) =	0 × (–2) =	0 × (–3) =
–1 × 3 =	–1 × 2 =	–1 × 1 =	–1 × 0 =	–1 × (–1) =	–1 × (–2) =	–1 × (–3) =
–2 × 3 =	–2 × 2 =	–2 × 1 =	–2 × 0 =	–2 × (–1) =		
–3 × 3 =	–3 × 2 =	–3 × 1 =	–3 × 0 =	–3 × (–1) =		–3 × (–3) =

Division grid

9 ÷ 3 =	6 ÷ 3 =	3 ÷ 3 =	0 ÷ 3 =	–3 ÷ 3 =	–6 ÷ 3 =	–9 ÷ 3 =
6 ÷ 2 =	4 ÷ 2 =	2 ÷ 2 =	0 ÷ 2 =	–2 ÷ 2 =	–4 ÷ 2 =	–6 ÷ 2 =
3 ÷ 1 =	2 ÷ 1 =	1 ÷ 1 =	0 ÷ 1 =	–1 ÷ 1 =	–2 ÷ 1 =	–3 ÷ 1 =
0 ÷ 0 =	0 ÷ 0 =	0 ÷ 0 =	0 ÷ 0 =	0 ÷ 0 =	0 ÷ 0 =	0 ÷ 0 =
–3 ÷ (–1) =	–2 ÷ (–1) =	–1 ÷ (–1) =	0 ÷ (–1) =	1 ÷ (–1) =	2 ÷ (–1) =	3 ÷ (–1) =
–6 ÷ (–2) =	–4 ÷ (–2) =	–2 ÷ (–2) =	0 ÷ (–2) =	2 ÷ (–2) =		
–9 ÷ (–3) =	–6 ÷ (–3) =	–3 ÷ (–3) =	0 ÷ (–3) =	3 ÷ (–3) =		9 ÷ (–3) =

N10 • Developing an exam question: number

Mathematical goals

To help learners to:

- use past examination questions creatively;
- select and use appropriate techniques and strategies to solve problems involving numerical, graphical and algebraic manipulation.

Starting points

Most learners will have solved problems such as these arithmetically, but they may not have considered alternative methods of solution.

Learners are provided with copies of an examination question similar to those found in GCSE examinations. They answer the question, then analyse the content and identify the skills required to obtain a correct solution. They develop the task by asking further questions and by changing the task in various ways. They develop their own examination questions and attempt to answer the questions designed by other learners.

Materials required

For each learner you will need:

- Sheet 1 – *Van hire*;
- Sheet 2 – *Car hire template (version 1)*;
- Sheet 3 – *Car hire template (version 2)*.

Time needed

About 1 hour.

Suggested approach

Beginning the session

Ask learners to work in pairs to tackle the GCSE examination questions in Sheet 1 – *Van hire*. When everyone has had time to have a go at this, hold a whole group discussion on the approaches used.

Whole group discussion (1)

(i) Completing the question

Collect suggestions for correct answers to each question and write on the board some of the approaches used.

For Question 1, some learners may have had difficulty in handling the decimals or in seeing that you can divide a smaller number by a larger one (such as 16 ÷ 50).

> How did you work out how much Hurt's vans cost per mile?
>
> (I did 1 600p divided by 50 on my calculator and got 32p.)
>
> (I did £32 divided by 100 miles and got £0.32 per mile.)
>
> Can you see any other ways of getting the answer from the table?

For Question 2, the working is quite complex and learners need to keep a careful record of what they are doing. It may be helpful to ask two learners to write out their solutions in full on the board and compare and contrast them.

(ii) Generating further questions

There are many other questions an examiner might have asked, based on this data. Invite learners to suggest some of these. When doing this, they should not seek to change the data in any way, but simply ask new questions based on the data.

As learners suggest possible further questions, list them on the board. For example, the following are all possibilities and are of varying difficulty.

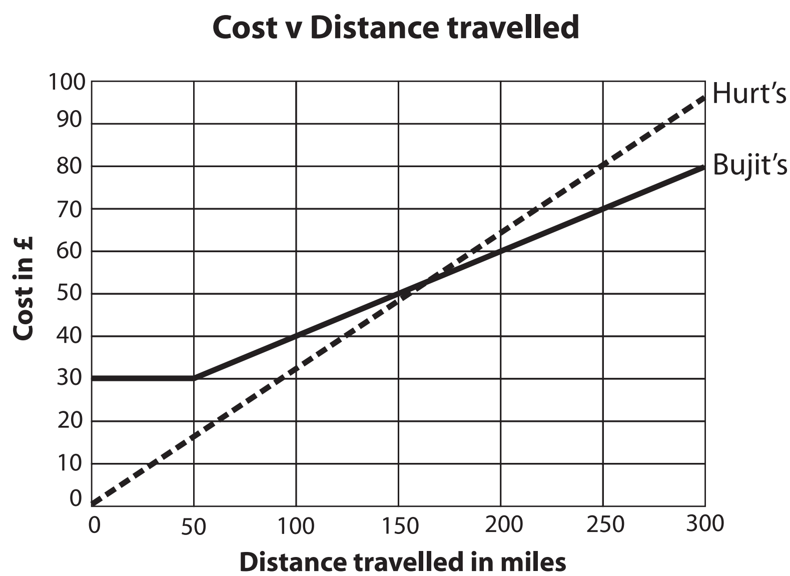
- Sanjay expected to travel 140 miles. Which company has the lower charge for this? How much would he save?
- Over what distance is Bujit's more expensive than Hurt's? Over what distance are Hurt's vans more expensive than Bujit's?
- Where is the cross-over point?
- Can you make a table showing Bujit's prices so that prices are easier to compare with Hurt's?

- Can you make a graph showing how the two companies' prices vary with the miles covered?
- Can you write a formula to show each company's prices?

Working in groups (1)

Ask learners to choose which of these further questions they think they can answer. Encourage them to work on them in pairs and try to agree on the correct answers. Learners may like to write their different solutions to the new questions on the board and compare them.

For example, the 'cross-over' point may be found by trial and error, or from a graph such as the one below:



Alternatively, they may use reasoning such as the following:

> After 50 miles, Hurt's are £14 cheaper. From then on the Hurt's rate is 12p per mile more expensive. £14 ÷ 12p = 117 approx, so, if he travels 117 more miles, then both will cost about the same. 117 + 50 = 167 miles.

or even by solving algebraic equations (though this is unlikely).

Whole group discussion (2)

(iii) Developing the situation

Hand out copies of Sheet 2 – *Car hire template (version 1)*. Ask learners to write their own GCSE question using this template.

Discuss with them how they might do this. They will need to decide on two costing systems to compare. They may be able to think of systems that are more realistic than the ones in the original GCSE question.

It is possible to ask more interesting questions if there is a cross-over point, so learners may like to try to make sure this happens.

They should then try to devise a few more questions. A range of questions is desirable, from easy to more difficult. The aim is to devise questions that the learners feel are challenging, but that they can answer correctly.

Answers should be written on the back of the sheet.

A second template (Sheet 3 – *Car hire template (version 2)*) is provided for learners who would like to devise a graphical question.

Working in groups (2)

The new questions can be passed around between groups to be answered by other learners. Where learners have difficulty, the question-writers should explain what they intended and act as teachers, helping other learners to answer the questions.

Alternatively, some of the new questions may be photocopied for future sessions or for homework.

Reviewing and extending learning

Finally, hold a whole group discussion on what has been learned, drawing out any common difficulties. You may wish to include a discussion of the level of difficulty of the new questions.

What learners might do next

Ask learners to choose another question from a past exam paper and follow the process adopted in this session.

(i) Do the question.

(ii) Ask new questions about the same situation (and answer them).

(iii) Change the situation and make up a new question.

Further ideas

This method for developing exam questions can be used in any topic. Examples in this pack include:

A8 Developing an exam question: generalising patterns;

S7 Developing an exam question: probability;

SS8 Developing an exam question: transformations.

N10 Sheet 1 – *Van hire*

Sanjay wants to hire a van to move some furniture.

He obtains the following information from two hire companies.

Bujit's Van Hire

£30 for the first 50 miles.

Every mile after that costs an extra 20p.

Hurt's Vans

You only pay for the miles you travel.

Miles travelled	50	100	150	200
Hire charge	£16	£32	£48	£64

1. How much do Hurt's vans cost per mile?

2. Sanjay expects to travel 175 miles.
   Which company has the lower charge for this distance?
   You must show all your working.

Think of some other questions the examiner might ask, based on this information.

N10 Sheet 2 – *Car hire template (version 1)*

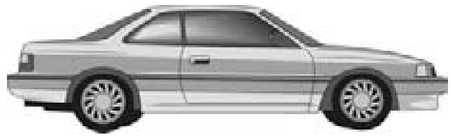
Cath wants to hire a car for a weekend.

She obtains the following information from two hire companies.

.... Car Hire

£ ....for the first
..............miles.
Every mile after that costs an extra .... p.

.... Car Hire



Miles travelled				
Hire charge				

N10 Sheet 3 – *Car hire template (version 2)*

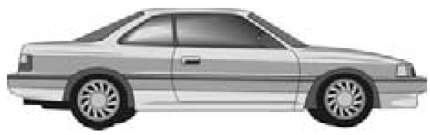
Cath wants to hire a car for a weekend.

She obtains the following information from two hire companies.

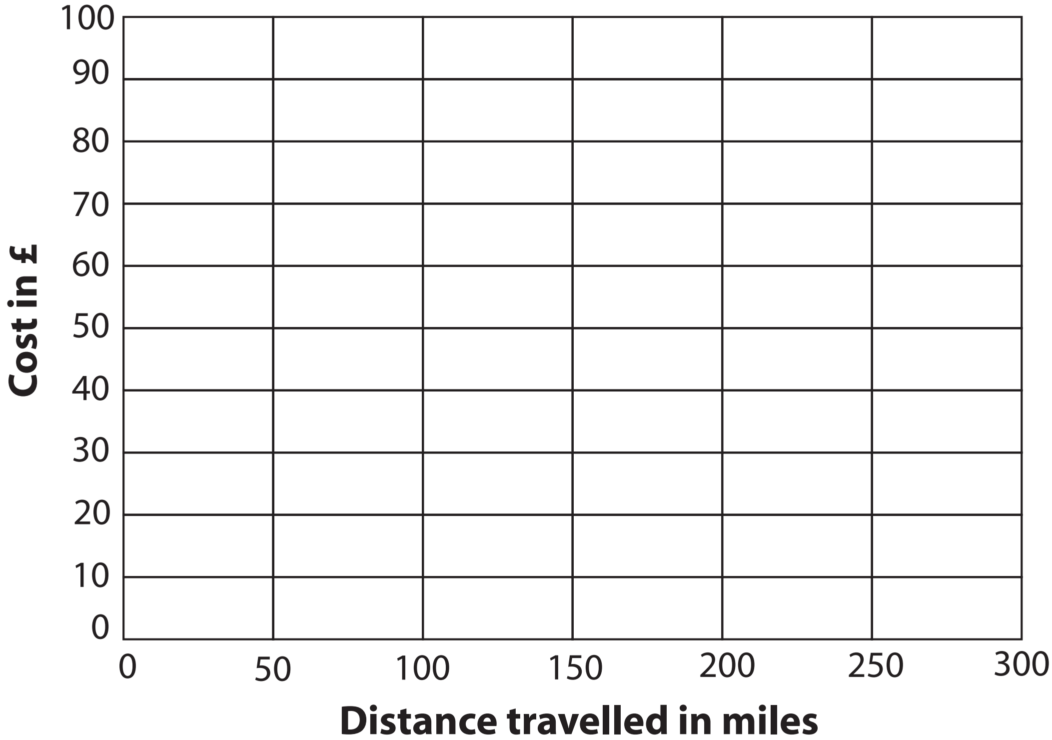
.... Car Hire

£ ....for the first
..............miles.
Every mile after that costs an
extra .... p.

.... Car Hire



Miles travelled				
Hire charge				



N11 • Manipulating surds

Mathematical goals

To enable learners to:

- identify equivalent surds.

To develop learners' ability to:

- simplify expressions involving surds.

Starting points

Learners should understand what a square root is and be able to remove brackets correctly.

Materials required

For each learner you will need:

- Sheet 1 – *True or false?*;
- mini-whiteboard.

For each small group of learners you will need:

- Sheet 2 – *Show that* (cut into strips);
- a selection from Card set A – *Equivalent surds*;
- Card set B – *Surd dominoes* (to make larger dominoes, this sheet could be enlarged onto A3 card).

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Give each learner a copy of Sheet 1 – *True or false?* and ask learners to use calculators to determine whether the statements in the first half of the sheet are true or false and then fill in the = or ≠ in the expressions at the bottom.

Whole group discussion (1)

Discuss the meaning of the $\sqrt{\ }$ sign and why we often leave it in expressions rather than using decimal approximation. Reinforce the generalisations at the bottom of Sheet 1 by linking with the numerical examples on the page.

Use mini-whiteboards to check some simple manipulations such as:

$$(\sqrt{2})^4 \qquad 3\sqrt{7} \times 2\sqrt{7} \qquad \sqrt{3}(\sqrt{2} - 6)$$

Working in groups (1)

Give each pair of learners one of the strips from Sheet 2 – *Show that.* Ask them to provide the working. As each pair finishes a strip, check their working and give them another. When all pairs have done at least three strips, give a selection of cards from Card set A *Equivalent surds* to each pair and ask them to discuss why each surd can be simplified in that way.

Whole group discussion (2)

Ask each pair of learners to explain one of their simplifications. Explain that 'expressing a surd in its simplest form' means that the number inside the surd should be as small as possible. Discuss strategies e.g. finding the highest factor that is a square number. Practise a few examples on mini-whiteboards.

Working in groups (2)

Give each pair of learners Card set B – *Surd dominoes* and ask them to make a domino chain. Note on the board any problems or difficulties that learners encounter.

For learners who find the manipulation of surds easy, some of the dominoes could be removed and replaced with blanks for them to complete. If any learners finish early they could be encouraged to add some dominoes of their own.

Reviewing and extending learning

Discuss together all the points that have been noted on the board. Then put the addition table and multiplication table below on the board for completion by learners working in pairs or on their own.

+		
	$3\sqrt{3} + 2\sqrt{2}$	$2\sqrt{3} + 3\sqrt{2}$
	$2\sqrt{3}$	$\sqrt{3} + \sqrt{2}$

×		
	$6\sqrt{6}$	18
	12	$6\sqrt{6}$

Learners could then write their own tables and pass them to other learners for completion.

What learners might do next

Use equivalent surds to simplify exact form solutions from quadratic equations.

Rationalise the denominator of surds.

Further ideas

The ideas in this session can be adapted for learning and practising a variety of skills, e.g. fractions, decimals, indices, logarithms and factorising.

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N11 Sheet 1 – *True or false?*

$\sqrt{9} + \sqrt{4} = \sqrt{13}$	$\sqrt{2} = 1.4142136$
$\sqrt{3} \times \sqrt{3} = 3$	$\sqrt{2} = 1.414213562$
$\sqrt{9} \times \sqrt{4} = \sqrt{36}$	$\sqrt{2} = 1.4142$
$\sqrt{2} \times \sqrt{2} \times \sqrt{2} = 2\sqrt{2}$	$\sqrt{9} - \sqrt{4} = \sqrt{5}$
$4\sqrt{3} + 5\sqrt{3} = 9\sqrt{3}$	$\dfrac{\sqrt{36}}{\sqrt{4}} = \sqrt{9}$
$2\sqrt{3} \times 4\sqrt{3} = 8\sqrt{3}$	

***a*, *b*, *x* and *y* are positive integers.**

Fill in = or ≠

$\sqrt{a} + \sqrt{b}$	$\sqrt{a + b}$	$\sqrt{x^2} + \sqrt{y^2}$	$x + y$
$\sqrt{a} - \sqrt{b}$	$\sqrt{a - b}$	$\sqrt{x^2 + y^2}$	$x + y$
$\sqrt{a} \times \sqrt{b}$	$\sqrt{ab}$	$\sqrt{x^2} \times \sqrt{y^2}$	xy
$\sqrt{\dfrac{a}{b}}$	$\dfrac{\sqrt{a}}{\sqrt{b}}$	$\sqrt{\dfrac{x^2}{y^2}}$	$\dfrac{x}{y}$

N11 Sheet 2 – *Show that*

Show that $\left(\sqrt{3} + 2\right)\left(\sqrt{3} + 4\right) = 11 + 6\sqrt{3}$

1

Show that $\left(\sqrt{5} - 2\right)\left(\sqrt{5} + 3\right) = \sqrt{5} - 1$

2

Show that $\left(2\sqrt{3} - 1\right)\left(2\sqrt{3} + 1\right) = 11$

3

Show that $\sqrt{18} \times \sqrt{2} - \sqrt{2} \times \sqrt{50} = -4$

4

Show that $\sqrt{3} + 2\left(3\sqrt{3} - 1\right) = 7\sqrt{3} - 2$

5

Show that $\left(\sqrt{3}\right)^2 + \left(3\sqrt{2}\right)^2 = 21$

6

N11 Card set A – *Equivalent surds*

$\sqrt{18} = 3\sqrt{2}$	$\sqrt{72} = 6\sqrt{2}$
$\sqrt{45} = 3\sqrt{5}$	$\sqrt{50} = 5\sqrt{2}$
$\sqrt{32} = 4\sqrt{2}$	$\sqrt{90} = 3\sqrt{10}$
$\sqrt{28} = 2\sqrt{7}$	$\sqrt{54} = 3\sqrt{6}$
$\sqrt{20} = 2\sqrt{5}$	$\sqrt{8} = 2\sqrt{2}$
$\sqrt{128} = 8\sqrt{2}$	$\sqrt{75} = 5\sqrt{3}$
$\sqrt{63} = 3\sqrt{7}$	$\sqrt{48} = 4\sqrt{3}$

N11 Card set B – *Surd dominoes*

Note: there are twenty dominoes in the set

$\sqrt{8}$	$\sqrt{18} + 3\sqrt{2}$	$3\sqrt{2}$	$\sqrt{90}$
$\sqrt{80}$	$\dfrac{\sqrt{50}}{5}$	$6\sqrt{2}$	$2\sqrt{3} \times 5\sqrt{3}$
$\sqrt{2}$	$\dfrac{\sqrt{72}}{\sqrt{3}}$	$12\sqrt{6}$	$\sqrt{40} \times \sqrt{90}$
$3\sqrt{10}$	$\dfrac{\sqrt{54}}{\sqrt{6}}$	60	Finish
40	$\dfrac{\sqrt{84}}{2}$	$2\sqrt{6}$	$\sqrt{8} + \sqrt{2}$
9	$\dfrac{8 + \sqrt{48}}{4}$	$5 + 2\sqrt{6}$	$\left(\sqrt{3}\right)^4$
30	$\sqrt{8} \times \sqrt{50}$	3	$\sqrt{128}$
Start	$\sqrt{3}\left(2\sqrt{3} - 1\right)$	20	$\sqrt{10} \times \sqrt{8}$
$2 + \sqrt{3}$	$3\sqrt{2} \times 4\sqrt{3}$	$8\sqrt{2}$	$2\sqrt{5} \times 4\sqrt{5}$
$6 - \sqrt{3}$	$2\sqrt{2}$	$\sqrt{21}$	$\left(\sqrt{3} + \sqrt{2}\right)^2$

N12 • Using indices

Mathematical goals

To introduce learners to:

- fractional and negative indices.

To enable learners to:

- evaluate numerical expressions using negative and fractional indices;
- use the rules of indices with integer and fractional powers of variables.

Starting points

Learners should have some knowledge of the rules of indices for multiplying and dividing numbers in index form.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Pairs activity* (two pages).

Each page should be photocopied onto different coloured card and cut up before the session.

For each pair of learners you will need:

- Card set B – *Indices* (two pages).

Time needed

At least 45 minutes.

Suggested approach

Beginning the session

Use mini-whiteboards to revise the laws of indices by asking for possible values for the question marks in:

$2^3 \times 2^4 = 2^?$

$3^7 \div 3^2 = 3^?$

$5^? \times 5^? = 5^{12}$

$2^? \div 2^? = 2^?$

Working in groups (1)

Write on the board a range of calculations in powers of 2, e.g.:

$16 \div 8 = 2$

$16 \times \frac{1}{2} = 8$

$8 \times \frac{1}{4} = 2$

$\sqrt{2} \times \sqrt{2} = 2$

$8 \times 4 = 32$

$16 \div \frac{1}{4} = 64$

$8 \div 8 = 1$

$\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$

You may need to explain cube roots.

Working in pairs, learners have to rewrite the calculations in powers of 2. They must work out any that they do not know, using the rules of indices. For example, for $16 \div \frac{1}{4} = 64$:

16 is 2^4; 64 is 2^6 so $\frac{1}{4}$ must be 2^{-2} so that $2^4 \div 2^{-2} = 2^6$.

Whole group discussion (1)

Invite learners to write one of their rewritten calculations on the board and say what it tells them about indices. Check that they understand negative, fractional and zero indices by asking them to evaluate some, using mini-whiteboards.

Working in groups (2)

Ask learners to work in groups of three or four. Give each group Card set A – *Pairs activity* (i.e. 16 cards in each colour).

Learners should place all the cards face down on the table. Learners take it in turns to pick up one card of each colour and turn them over so that partners can see them. If they match, the learner keeps the pair. If they do not match, the learner places them face down on the table again. If a learner claims a pair that does not match and is correctly challenged by another member of the group, they have to put the cards back and miss a turn. The winner is the learner who has the most pairs.

Whole group discussion (2)

When the activity is finished ask each learner to explain to the whole group why they matched one of their pairs.

Working in groups (2)

Ask learners to work in pairs. Give out Card set B – *Indices* to each pair. These cards move the learning on from numerical indices to algebraic indices. Ask learners to find at least three pairs of equivalent cards but encourage them to find more and give them enough time to do so. Write on the board card-pairs that the learners have found and ask for explanations of why they are equivalent.

Next, ask learners to find sets of cards such that one card is the product of the rest. Most will find two that multiply to make the third but phrasing the task in this way allows learners who are finding it easy to be more adventurous. Again, ask for explanations.

Next, ask for sets of three cards such that the first divided by the second is equal to the third. Discuss these.

Reviewing and extending the learning

Put a number '8' in the middle of the board. Invite the whole group to suggest different numbers in index form that are equal to 8. Products and quotients can also be allowed.

Give each pair of learners a number to work on, say 9, and then share all the ideas with the whole group.

Finally, repeat the process for x^2 or some other power of x.

What learners might do next

Learners could work on differentiation and integration of functions with negative or fractional powers of x.

Further ideas

Cards can be used in this way for a range of topics that require only mental calculations e.g. fractions, surds and names of shapes.

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N12 Card set A – *Pairs activity* (page 1)

$8^{\frac{2}{3}}$	$(-1)^{\frac{1}{3}}$	$9^{\frac{3}{2}}$	$16^{\frac{1}{4}}$
$\left(\frac{1}{2}\right)^{-3}$	$\left(\frac{2}{5}\right)^{-2}$	$\left(\frac{1}{9}\right)^{-\frac{1}{2}}$	$36^{-\frac{1}{2}}$
3^{-2}	$\left(\frac{16}{25}\right)^{\frac{3}{2}}$	3^{-1}	$4^{-\frac{3}{2}}$
$8^{-\frac{1}{3}}$	$(-1)^{-2}$	$\left(\frac{1}{5}\right)^{-1}$	$\left(\frac{4}{9}\right)^{\frac{1}{2}}$

N12 Card set A – *Pairs activity* (page 2)

$\frac{2}{3}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{64}{125}$
$\frac{1}{3}$	4	1	$\frac{1}{2}$
8	–1	$\frac{1}{9}$	3
5	$\frac{25}{4}$	27	2

N12 Card set B – *Indices* (page 1)

x^{-3}	x^{3}	x^{2}	$\sqrt{x}$
$x^{\frac{3}{2}}$	$x^{\frac{1}{2}}$	$\sqrt{x^{3}}$	$x\sqrt{x}$
$x^{-\frac{1}{2}}$	$2x^{-2}$	$\dfrac{\sqrt{x}}{x}$	x^{4}
$\dfrac{1}{2}x^{-\frac{1}{2}}$	$\dfrac{1}{2}x^{-2}$	$x^{\frac{1}{3}}$	x

N12 Card set B – *Indices* (page 2)

$\sqrt[3]{x}$	x^{-1}	$2x^{-1}$	$2\sqrt{x}$
$x^{-\frac{1}{3}}$	x^{-2}	$\frac{1}{2x^2}$	$x^{\frac{2}{3}}$
x^{-4}	$\frac{1}{2\sqrt{x}}$	x^0	$x^{-\frac{3}{2}}$
$\frac{2}{\sqrt{x}}$	$\frac{1}{x}$	$\frac{1}{2x}$	$\sqrt[3]{x^2}$

N13 • Analysing sequences

Mathematical goals

To enable learners to:

- define a sequence using the general form of the nth term, e.g. $u_n = 2n^3$;
- define a sequence inductively, e.g. $u_{n+1} = 4u_n - 1$;
- recognise and define an arithmetic progression;
- recognise and define a geometric progression;

and to reflect on and discuss these processes.

> These goals may be adapted for learners aiming at lower level qualifications. For example, you could leave out explicit references to arithmetic and geometric progressions.

Starting points

Learners should have some understanding of what a number sequence is.

Materials required

For each learner you will need:

- mini-whiteboard;
- Sheet 1 – *Arithmetic and geometric progressions*.

For each small group of learners you will need:

- Card set A – *Sequences* (two pages);
- Card set B – *Blanks*.

Time needed

At least 45 minutes.

Suggested approach

Beginning the session

Explain that learners have to make as many number sequences as they can, using the numbers on the *Sequences* cards they will be given. Each sequence must contain at least three cards.

> Learners aiming at lower levels can be asked to read out one of their descriptions. Then other members of the group have to write down the sequence (if possible) from the description and compare it with the original. As a result, descriptions may need to be rewritten.

Working in groups (1)

Ask learners to work in pairs. Give out Card set A – *Sequences*.

Learners who find it easy to make sequences can be encouraged to create more complex sequences.

When plenty of sequences have been created, put some blank cards out on each table. For each sequence that learners have made (or a selection if they have a lot), ask them to copy the sequence onto a blank card and to add a description, in words, of the pattern of the sequence.

Whole group discussion (1)

Write some of the sequences on the board and discuss how to express them using mathematical notation, in particular u_n as the nth term.

For example,

"1, 5, 9 ⇒ add 4 on each time" becomes:

$$u_1 = 1 \quad u_2 = u_1 + 4 \quad u_3 = u_2 + 4 \quad \ldots \quad u_{n+1} = u_n + 4$$

or

"1, 4, 9 ⇒ square numbers" becomes:

$$u_1 = 1^2 \quad u_2 = 2^2 \quad u_3 = 3^2 \quad \ldots \quad u_n = n^2$$

Using mini-whiteboards, practise the definitions. Give learners an algebraic or an inductive definition and ask them to write down the first three terms, or the fifth term, etc. Giving an inductive definition without a starting term can highlight the need for a starting number when defining a sequence inductively.

Working in groups (2)

Give the pairs of learners some time to write inductive or nth term definitions for the sequences they have on their cards. At this stage, most of their definitions will probably be inductive so you can encourage them to write alternative algebraic definitions.

Whole group discussion (2)

Ask learners to write an algebraic or inductive definition of one of their sequences on the board and invite the rest of the group to give the sequence from that definition. This can then be compared with the original sequence to see if the definition is correct.

Define arithmetic and geometric progressions. Ask learners to go back to Card set A. Ask learners to:

- find an arithmetic progression that includes the number 20;
- find a geometric progression that includes the number 3;
- find the longest arithmetic progression that you can;
- find the longest geometric progression that you can.

Write some of the answers on the board and ask learners to find algebraic and inductive definitions for them.

Reviewing and extending learning

Ask each learner to complete Sheet 1 – *Arithmetic and geometric progressions*. When they have finished, they should swap with a partner who should comment on their progressions and explanations.

Learners can generalise their learning to obtain $u_n = a + (n - 1)d$ and $u_n = ar^{n-1}$ for arithmetic and geometric progressions respectively.

What learners might do next

Give at least one reason why each of the following sequences might be the odd one out, i.e. it has a property that none of the others has.

(a) 2, 8, 18, 32 . . .

(b) 2, 5, 8, 11 . . .

(c) 3, 6, 12, 24 . . .

(d) 2, 1, 0.5, 0.25 . . .

(e) 2, –6, 18, –54 . . .

For example: '(b) is an arithmetic sequence and the others are not.'

Give possible arithmetic progressions such that the sum of the first five terms is 260.

This session could be followed by work on sums of series.

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N13 Card set A – *Sequences* (page 1)

40	0.7	12	$\frac{1}{2}$
−2	18	12.5	27
0.9	6	54	25
1	15	4	3
−0.1	−3	20	8

N13 Card set A – *Sequences* (page 2)

$\frac{3}{2}$	$\frac{1}{4}$	36	$\frac{2}{3}$
–6	0.2	–20	–0.5
$\frac{1}{3}$	–10	$\frac{1}{6}$	$\frac{3}{4}$
60	$\frac{1}{8}$	37.5	17
9	0.1	51	0.3

N13 Card set B – *Blanks*

N13 Sheet 1 – *Arithmetic and geometric progressions*

Name: . . . .

Write down the first four terms of an arithmetic progression that has the number 100 as its second term.

Explain why it is an arithmetic progression.

Write down an inductive definition for this sequence.

Write down an algebraic definition for this sequence.

Write down the first four terms of a geometric progression that has the number 100 as its third term.

Explain why it is a geometric progression.

Write down an inductive definition for this sequence.

Write down an algebraic definition for this sequence.

A1 • Interpreting algebraic expressions

Mathematical goals

To help learners to:

- translate between words, symbols, tables, and area representations of algebraic expressions;
- recognise the order of operations;
- recognise equivalent expressions;
- understand the distributive laws of multiplication and division over addition (expansion of brackets).

Starting points

This unit develops the ideas presented in **N5 Understanding the laws of arithmetic**. Learners will need to recall how to find the area of simple compound shapes made from rectangles and simple indices, and to understand the difference between 3×2 and 3^2.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Algebraic expressions*;
- Card set B – *Explanations in words*;
- Card set C – *Tables of numbers*;
- Card set D – *Areas of shapes*;
- glue stick;
- felt tip pens;
- large sheet of paper for making a poster.

Time needed

From 1 to 2 hours. If your sessions are normally shorter than this, you can split this session into two.

Suggested approach

Beginning the session

Hold a short question and answer session, using mini-whiteboards, to revise how the order of operations is represented algebraically.

> Show me an algebraic expression that means:
>
> Multiply n by 3, then add 4.
> Add 4 to n, then multiply your answer by 3.
> Add 2 to n, then divide your answer by 4.
> Multiply n by n, then multiply your answer by 4.
> Multiply n by 6, then square your answer.

Working in groups (1)

Arrange learners in pairs or threes.

Give each group Card set A – *Algebraic expressions* and Card set B – *Explanations in words*. Ask learners to take turns at matching cards and explaining their reasons for each matching. Point out that some cards are missing. Learners will need to make these extra cards themselves.

The activity is designed to help learners interpret the symbols and realise that the symbolism defines the order of operations. Some learners may notice that some expressions are equivalent, e.g. $2(n + 3)$ and $2n + 6$. Do not comment on this at this stage.

Now give learners Card set C – *Tables of numbers* and ask them to match these to Card sets A and B. Some tables have numbers missing. Learners will need to work these out.

Learners will soon notice that there are fewer tables than algebraic expressions. This is because some tables match more than one expression. Allow learners time to discover this for themselves.

This activity is designed to encourage learners to substitute into expressions and thus, again, to interpret their meaning. At this stage, they will begin to notice that several expressions are equivalent, but they may not realise why.

For learners who finish quickly, ask them to find out if the pairs of expressions always give the same answer, even when fractions or decimals are substituted. Push them further to explain why these pairs match for all numbers.

Encourage those who struggle to substitute numbers for letters in the algebraic expressions.

Reviewing and extending learning (1)

Ask learners to suggest reasons why different expressions always appear to give the same answer. (You don't need to provide answers yourself at this stage).

Can you generate additional examples of your own?

Volunteers may like to offer suggestions on the board.

Leave 'open' the question of why expressions are equivalent. The next part of the session will take these ideas further, so they do not need to be resolved at this point.

(If you run this over two sessions, this would be a good point to break off.)

Working in groups (2)

Begin this part of the session by considering just the *Algebraic expressions* cards (Set A). Ask learners to try to remember which expressions went together/were equivalent. If they have forgotten, they should try substituting some numbers of their own for n. This will encourage them to interpret the expressions without the help of the *Explanations in words* cards (Set B).

Now hand out Card set D – *Areas of shapes*. Ask learners to match these with the cards in Set A – *Algebraic expressions*. When learners reach agreement, they should paste their cards onto a large sheet of paper, to make a poster. They should write on the poster why the areas show that different algebraic expressions are equivalent. These posters may then be displayed for the final whole group discussion.

Reviewing and extending learning (2)

Hold a whole group discussion to review what has been learned during this session. Ask pairs of learners to present their posters.

Use mini-whiteboards and questioning to begin to generalise the learning.

Draw me an area that shows this expression: $3(x + 4)$.
Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(4y)^2$.
Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(z + 5)^2$.
Write me a different expression that gives the same area.

Draw me an area that shows this expression: $\frac{w+6}{2}$.

Write me a different expression that gives the same area.

What rules have you found for rearranging expressions?

What learners might do next

Ask learners if they can draw diagrams that might show a set of expressions containing subtractions, such as the following:

$2(x-3)$

$(x+3)(x-3)$

x^2-9

$(x-3)^2$

x^2-6x+9

Learners might contribute their own questions as a homework activity and evaluate the work of other learners as a follow-up.

Further ideas

This activity uses multiple representations to deepen understanding of algebra. This type of activity may be used in any topic where a range of representations is used. Examples in this pack include:

SS6 Representing 3D shapes;

SS7 Transforming shapes;

S5 Interpreting bar charts, pie charts, box and whisker plots;

S6 Interpreting frequency graphs, cumulative frequency graphs, box and whisker plots.

A1 Card set A – *Algebraic expressions*

E1 $\dfrac{n+6}{2}$	E2 $3n^2$
E3 $2n + 12$	E4 $2n + 6$
E5 $2(n + 3)$	E6 $\dfrac{n}{2} + 6$
E7 $(3n)^2$	E8 $(n + 6)^2$
E9 $n^2 + 12n + 36$	E10 $\dfrac{n}{2} + 3$
E11 $n^2 + 6$	E12 $n^2 + 6^2$
E13	E14

A1 Card set B – *Explanations in words*

W1 **Multiply n by two, then add six.**	W2 **Multiply n by three, then square the answer.**
W3 **Add six to n, then multiply by two.**	W4 **Add six to n, then divide by two.**
W5 **Add three to n, then multiply by two.**	W6 **Add six to n, then square the answer.**
W7 **Multiply n by two, then add twelve.**	W8 **Divide n by two, then add six.**
W9 **Square n, then add six.**	W10 **Square n, then multiply by nine.**
W11	W12
W13	W14

A1 Card set C – *Tables of numbers*

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

n	1	2	3	4
Ans		4		5

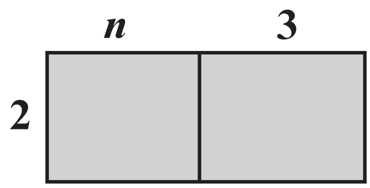
T8

n	1	2	3	4
Ans	6.5	7	7.5	8

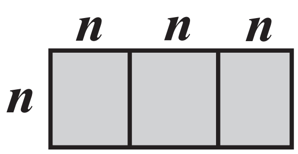
A1 Card set D – *Areas of shapes*

A1

A2



A3



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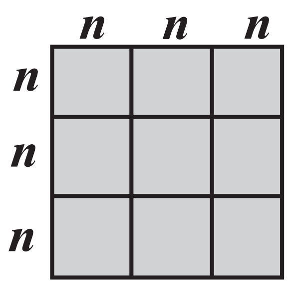
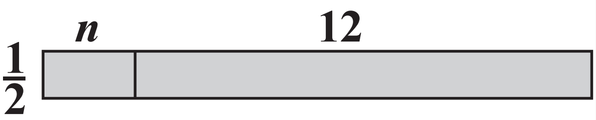


A5

A6



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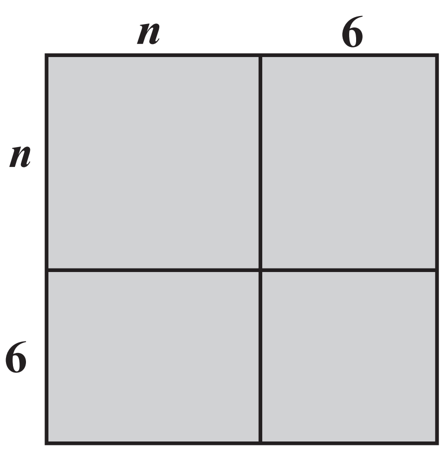

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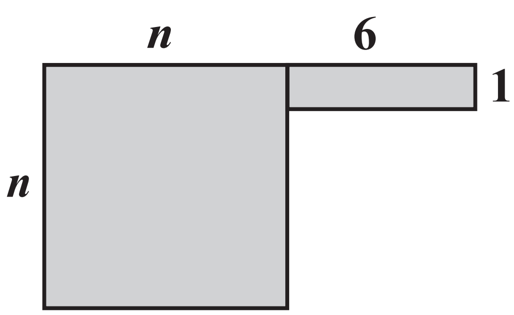

A2 • Creating and solving equations

Mathematical goals

To enable learners to:

- create and solve their own equations, where the unknown appears once;
- develop confidence with the notation used in equations.

To help learners to:

- teach and learn from each other.

Starting points

Most learners will have been taught rules for solving equations such as 'change the side, change the sign' or 'you always do the same to both sides'. When used without understanding, such rules result in many errors. For example:

$3x = 15 \Rightarrow x = \frac{15}{-3}$ (change the side, change the sign);

$3(x - 2) = 6 \Rightarrow x - 2 = 3$ (taking 3 from both sides).

'Doing the same to both sides' is the more meaningful method, but there are two difficulties:

- knowing how to change both sides of an equation so that equality is preserved;
- knowing which operations lead towards the desired goal.

Building equations is easier than solving them because it postpones the second difficulty and so is an easier place to start.

It is helpful if learners have already encountered the following ideas.

- Addition is the inverse of subtraction (and vice versa).
- Multiplication is the inverse of division (and vice versa).
- The use of brackets/fraction bars when multiplying and dividing.

Materials required

For each learner you will need:

- Sheet 1 – *Creating equations*;
- Sheet 2 – *Solving equations*.

It is also helpful to make OHTs of Sheets 1 and 2 for use in the whole group introduction.

Time needed

About 1 hour.

Suggested approach

Beginning the session

1. Build an equation

Write down a letter and its value on the board, e.g. $x = 3$. (This may be done on an OHT of Sheet 1.)

Using learners' suggestions for operations, build up an equation, step by step, using each of the four rules, +, –, ×, ÷ and whole numbers between 1 and 10.

$$x = 3$$

Add 5

$$x + 5 = 8$$

Divide by 4

$$\frac{x + 5}{4} = 2$$

Subtract 1

$$\frac{x + 5}{4} - 1 = 1$$

Multiply by 3

$$3\left(\frac{x + 5}{4} - 1\right) = 3$$

As learners suggest each operation, you supply the notation and explain it carefully. For example, explain that we use brackets to show that a whole expression is being multiplied and that we use the fraction bar rather than the usual division symbol (÷).

During this process, experience has shown that it is better not to simplify the left hand side of the equation at any stage. For example, if learners suggest the four operations +5, –1, ÷4, ×3, then we write $3\left(\frac{(x + 5) - 1}{4}\right)$ rather than $3\left(\frac{x + 4}{4}\right)$.

2. Check the equation

Ask the group to check that the original value of x still satisfies the final equation.

$$3\left(\frac{3+5}{4}\right) - 1 = 3\left(\frac{8}{4} - 1\right) = 3(2-1) = 3 \times 1 = 3$$

3. Solve the equation

Hide all the steps except the final equation and ask the group to recall each operation in sequence.

> This equation tells the story of 'a day in the life of x'.
> What happened to it first? How can you tell by looking only at the equation?
> What then?
> What then?
> What was the last thing that happened?

In this way, show that the final equation tells the story of the operations used.

> Suppose you had started with this equation and you wanted to find the value of x.
> How could you do this?
> How can you undo what we have just done?
> You take your socks and boots off in the reverse order to the order you put them on.
> It's the same here.

Gradually get the group to unpick each step in reverse order. As they do this, uncover the preceding equations one by one and write the corresponding operation to the right of each equation (with upward arrows):

	$x = 3$	
Add 5		Subtract 5
	$x + 5 = 8$	
Divide by 4		Multiply by 4
	$\dfrac{x+5}{4} = 2$	
Subtract 1		Add 1
	$\dfrac{x+5}{4} - 1 = 1$	
Multiply by 3		Divide by 3
	$3\left(\dfrac{x+5}{4}\right) - 1 = 3$	

You will probably need to work through one or two more examples like this with the group, until they get the idea. It is worth changing the letter used (from x) each time, just to make the point that there is nothing special about it.

4. Create your own equation

Ask learners to create two equations of their own in a similar way. Sheet 1 – *Creating equations* provides a structure for this. After creating each equation, learners should check that it works by substituting the answer back into it.

Learners who struggle may be asked to restrict themselves to fewer steps and operations to start with.

When learners are satisfied that their equations work (and maybe when they have checked them with you), ask them to write the equations on Sheet 2 – *Solving equations*.

Working in groups

Each learner should then give their Sheet 2 to a partner. The partner should try to 'undo' the operations, step by step. Partners may call on originators for help if they get stuck. Encourage learners to help each other as much as possible.

As learners get the idea, the structured sheets may be discarded and learners may enjoy creating more challenging equations. Encourage them to do this by having more steps to the equation, rather than by using harder numbers.

Reviewing and extending learning

Ask learners to write their 'favourite creations' on the board and ask other learners, working in pairs, to solve them.

Ask learners to use their mini-whiteboards to write down algebraic equations that correspond to some 'think of a number' problems. For example, you might say:

Think of a number, call it n.
Double it.
Add 4.
Divide your answer by 7.
Multiply your answer by 2.
The result is 4.
Show me the equation.

And your learners might respond: $2\left(\frac{2n + 4}{7}\right) = 4$

What learners might do next

Learners may enjoy introducing further operations, such as $\frac{1}{x}$ (the inverse operation) and +/– (the 'change the sign' operation). Both are self-inverses.

Using these, learners may create more complex equations, such as $1 - \frac{1}{n - 3} = \frac{1}{2}$.

This was created by starting with $n = 5$ and then operating as follows:

–3, $\frac{1}{x}$ (invert), +/– (change the sign), +1.

To undo this sequence, we simply do:

–1, +/– (change the sign), $\frac{1}{x}$ (invert), +3.

In addition, you may like to use **A3 Creating and solving harder equations**. This considers equations where the unknown appears more than once.

Further ideas

This kind of activity may be used for other 'doing' and 'undoing' activities.

For example:

- Learners might draw a rectangle and then calculate the area and perimeter. Partners must then attempt to 'find' the original rectangle from the area and perimeter.

- Learners might write down an expression involving brackets and then expand them. Partners must then attempt to factorise the resulting expansion to get back to the original expression.
- Learners might sketch a distance–time graph and then write a story describing what happened. Partners must then try to reconstruct the graph from the story.

A2 Sheet 1 – *Creating equations*

Name . . . .

1. Create two equations in the spaces below.
   Show, step by step, how you make your equations, by writing an operation next to each arrow.

Operations

$x =$

....

....

....

....

This is equation 1

Operations

$y =$

....

....

....

....

This is equation 2

2. Check that your equations work by substituting the original value.

Check equation 1

Check equation 2

A2 Sheet 2 – *Solving equations*

3. Rewrite your final equations in the spaces below.
   Give this sheet to your partner and ask them to solve your equations.

Name of partner . . . .

4. Solve the equations, showing what you do at each step.

Operations	**Equation 1**
	....
....	....
....	....
....	....
....	....

Operations	**Equation 2**
	....
....	....
....	....
....	....
....	....

5. Correct your partner's work. Did they follow the steps and solve the equation correctly? Describe any difficulties that they had.

Comments

A3 • Creating and solving harder equations

Mathematical goals

To enable learners to:

- create and solve equations, where the unknown appears more than once.

To help learners to:

- recognise that there may be more than one way of solving such equations;
- teach and learn from each other.

Starting points

Before embarking on this session, we strongly suggest that you use **A2 Creating and solving equations**. This will ensure that learners are able to solve equations where the unknown appears only once.

Materials required

For each learner you will need:

- Sheet 1 – *Creating equations*;
- Sheet 2 – *Solving an equation in different ways.*

It is helpful to make OHTs of Sheets 1 and 2 for use in the whole group introduction.

Optional:

- computer programs *Balance 1, Balance 2* and *Cover up;*
- Sheet 3 – *Instructions for using the program* Balance 1;
- Sheet 4 – *Instructions for using the program* Balance 2;
- Sheet 5 – *Instructions for using the program* Cover up.

Time needed

About 1 hour.

Suggested approach

Beginning the session

1. Build an equation

Write a letter and its value (e.g. $x = 2$) on the board. You may wish to use an OHT of Sheet 1 – *Creating equations*.

Build up an equation, step by step, as in **A2 Creating and solving equations**. This time, however, include a step that uses the letter itself. For example, you might add $3x$ to both sides:

$x = 2$

Add $3x$

$4x = 2 + 3x$

Add 6

$4x + 6 = 8 + 3x$

2. Check the equation

Ask learners to check that the original value for x still fits the final equation.

Thus: $4 \times 2 + 6 = 14$; $8 + 3 \times 2 = 14$.

3. Solve the equation

Now, starting with $4x + 6 = 8 + 3x$, ask how can we go back and discover the value of x. Ask learners to suggest alternative methods and explore each one. Follow up dead ends as necessary and ask them to explain why they do not lead to a solution.

For example:

$4x + 6 = 8 + 3x$

Subtract 6

$4x = 2 + 3x$

Subtract $3x$

$x = 2$

$4x + 6 = 8 + 3x$

Subtract $3x$

$x + 6 = 8$

Subtract 6

$x = 2$

$4x + 6 = 8 + 3x$	$4x + 6 = 8 + 3x$
Subtract 8	Subtract $4x$
$4x - 2 = 3x$	$6 = 8 - x$
Subtract $3x$	Subtract 8
$x - 2 = 0$	$-2 = -x$
Add 2	Multiply by -1
$x = 2$	$2 = x$

4. Create your own equation

Give each learner Sheet 1 and Sheet 2 – *Solving an equation in different ways*.

Ask learners to use Sheet 1 to:

- build an equation, operating on both sides with the unknown at some stage;
- check the equation, by substitution;
- solve the equation using a different method from the one used to create it.

Working in groups

Learners should now copy their equations onto Sheet 2 and give this sheet to their partner. The partner should try to solve the equation in different ways. The originators should try to help the partners when they get stuck.

Encourage learners to help each other as much as possible. As learners get the idea, the structured Sheets 1 and 2 can be discarded and learners may enjoy creating more challenging equations.

This activity enables learners at all levels to create equations of varying degrees of difficulty. However, all learners should be encouraged to be adventurous.

Reviewing and extending learning

Ask learners to write their 'favourite creations' on the board and invite other learners to solve them. See if several ways can be found for solving each equation.

Ask learners to use their mini-whiteboards to write down algebraic equations that correspond to some 'think of a number' problems. This time, include some that involve 'the number you first thought of'.

> Think of a number, call it n.
> Multiply it by 7.
> Add 4.
> Subtract the number you first thought of.
> The result is 34.
> Show me the equation.
> Now solve it.

What learners might do next

Further, individual practice at solving equations may be done using the computer software programs *Balance 1, Balance 2* and *Cover up* (available on the DVD-ROM/CD).

Balance 1 asks learners to say what operations they must do in order to solve an equation. The computer then carries out these operations. Learners do the strategic thinking, while the computer does the manipulation.

Balance 2 is the same as *Balance 1*, except that learners have to identify appropriate operations to perform, and then carry out these operations themselves.

Cover up uses a different strategy for tackling equations. At each step, the learner 'covers up' part of the equation and reasons what number must be hidden.

All three programs also allow the user to create equations.

Instructions for using these programs are given in Sheets 3, 4 and 5.

Learners may like to create examples where both sides of the equation are multiplied or divided by some expression containing the unknown.

For example:

$x = 4$

Divide by x ↓

$1 = \frac{4}{x}$

Add 8 ↓

$9 = \frac{4}{x} + 8$

$x = 4$

Subtract 5 ↓

$x - 5 = -1$

Square ↓

$(x - 5)^2 = 1$

When unpicking the second equation, learners should notice that the natural thing to do is to square root both sides and obtain the positive root, thus obtaining $x = 6$ as a solution. Check that $x = 4$ and $x = 6$ both satisfy the equation. Point out that squaring may introduce a second solution because, when we square root, we can either follow the positive or negative root. We have to do both to find all the solutions.

You may find that learners quickly begin to generate quadratic equations that they cannot yet solve. This is a good springboard for a future session.

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A3 Sheet 1 – *Creating an equation*

Name . . . .

1. Create an equation in the space below. Remember to show every step.

Operations

$x =$

....

....

....

2. Check that your equation works by substituting the original value of x.

Check

3. Solve your own equation. Use a different method from the one you used to create it.

Solution

A3 Sheet 2 – *Solving an equation in different ways*

1. Write your equation from Sheet 1 in the space below.

Equation

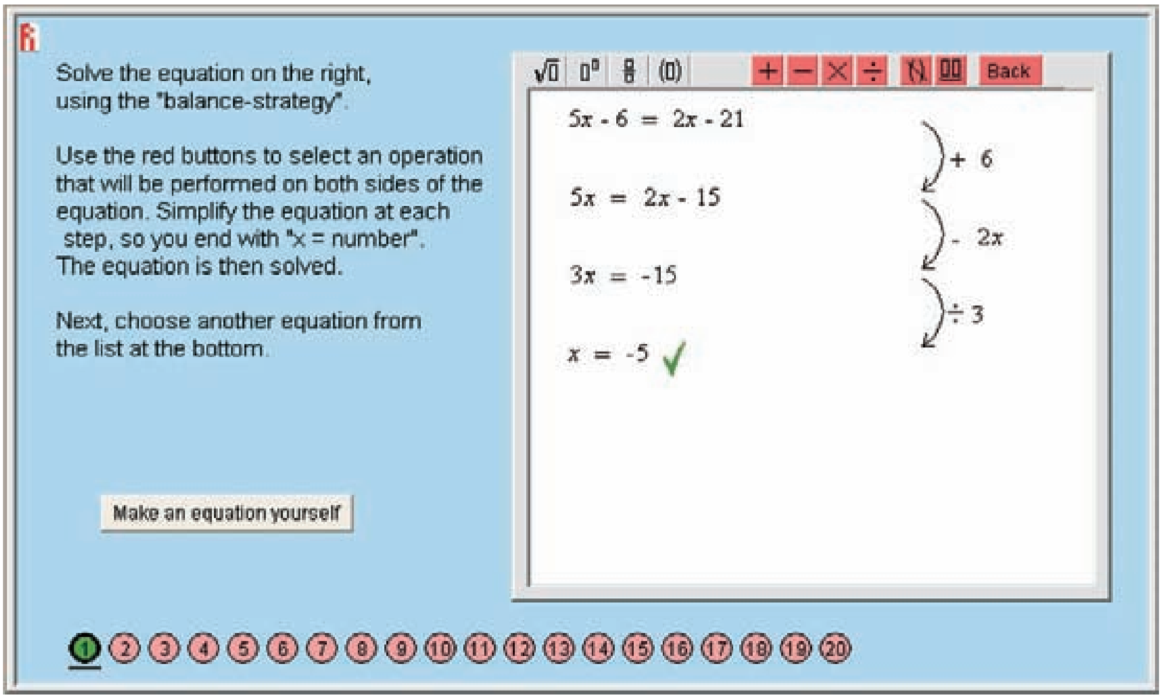
2. Give this sheet to your partner and ask them to solve your equation in as many different ways as possible.

Name of partner . . . .

Solution 1	Solution 2
Solution 3	Solution 4

3. Correct your partner's work. Did they do it correctly? Talk to your partner about any errors you found and write some comments to help.

A3 Sheet 3 – *Instructions for using the program* Balance 1



Purpose

To give practice at solving linear equations with one unknown in which the unknown appears on either side or on both sides of the equation, using the balance method. The interesting feature in *Balance 1* is that the user does the strategic thinking, while the computer does the manipulation.

Running the program

You are given (or you create) a series of equations. You must decide what has to be done to each side of an equation in order to simplify and eventually solve it. Your decisions are entered using the red buttons. From left to right, these are: *add, subtract, multiply, divide, expand brackets* (use the distributive law), and *combine alike terms*. If you change your mind at any stage, you can use the *Back* button.

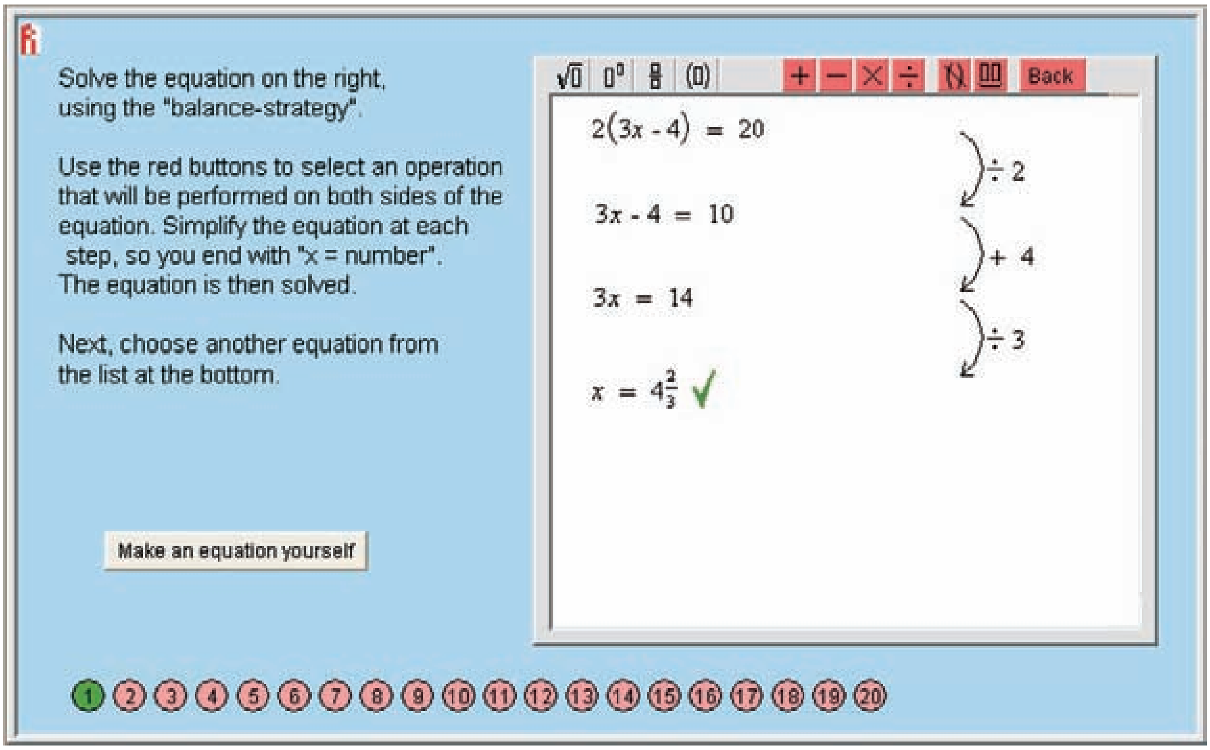
For example, here is one way to solve the equation $5x - 6 = 2x - 21$ using this program.

- Click on the red [+] button.
- Type '6' in the rectangular box on the right and hit 'Enter'.
- Click on the red [–] button.
- Type '$2x$' in the rectangular box on the right and hit 'Enter'.
- Click on the red [÷] (divide) button.
- Type '3' in the rectangular box on the right and hit 'Enter'.
- The solution $x = -5$ appears on the screen followed by a tick indicating that the solution is correct.

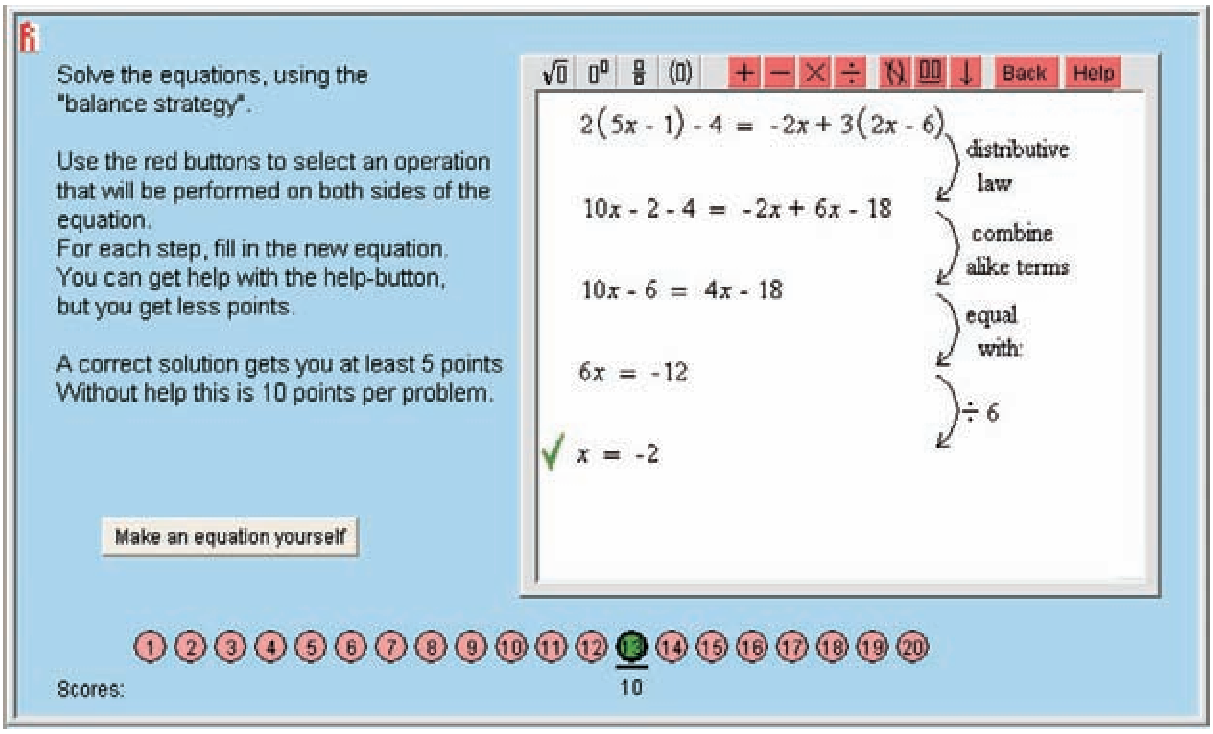
A3 Sheet 3 – *Instructions for using the program* Balance 1 (continued)

Along the bottom of the screen there are 20 pink circles indicating that there are twenty equations that you can solve. After you have correctly solved an equation, the colour of the circle changes to green.

If you wish, you can make your own equations. Click the 'Make an equation yourself' button, then enter your equation in the small drop-down window. Click the red [+] button. Your equation will now appear on the right hand side of the screen, where it can be solved as before. Equations that cannot be solved will not appear.



A3 Sheet 4 – *Instructions for using the program* Balance 2



Purpose

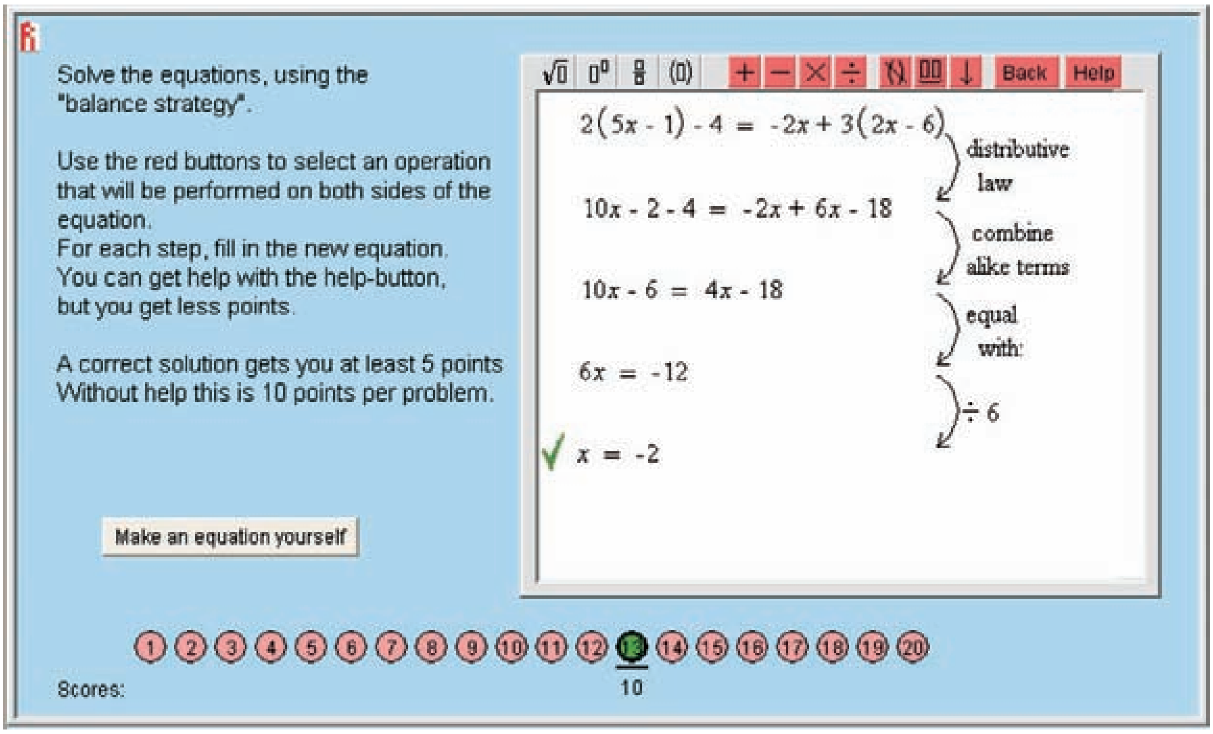
To give practice at solving linear equations with one unknown in which the unknown appears on either side or on both sides of the equation, using the balance method.

Balance 2 is more demanding than *Balance 1*, in that you not only need to decide what must be done to each side at each step, you must also type in the result of doing this.

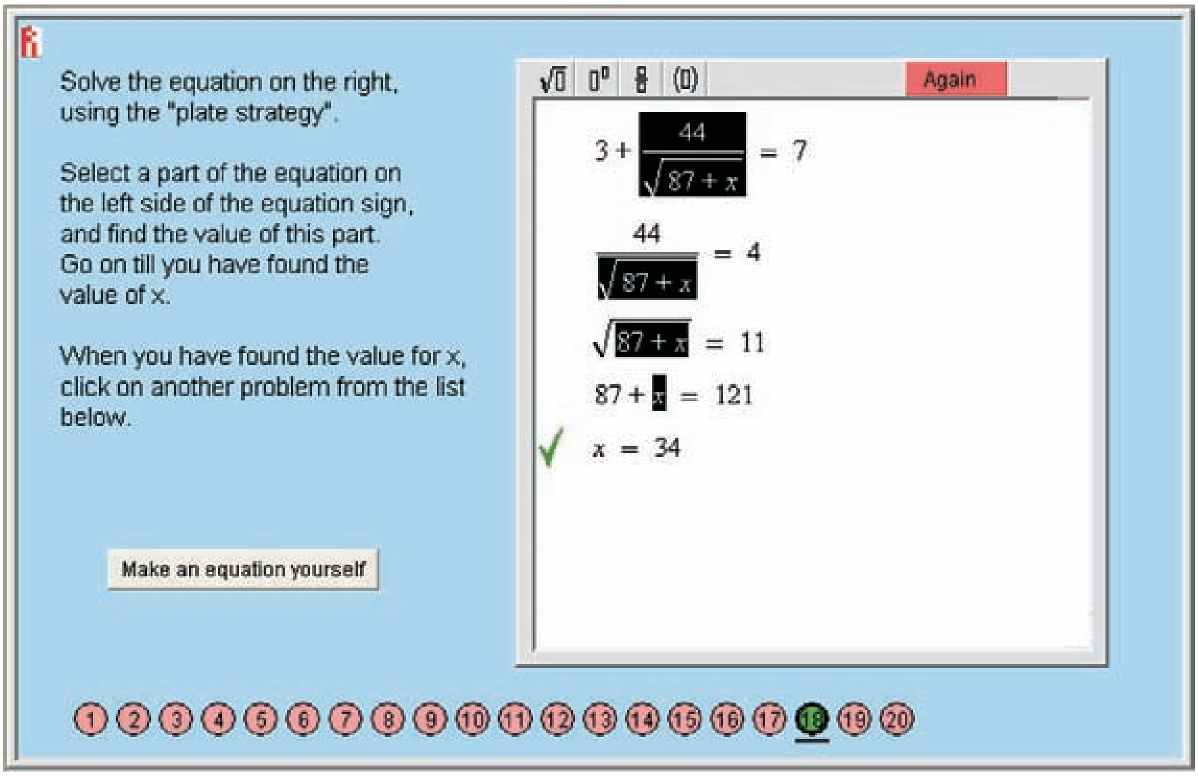
Running the program

When you have decided what to do for a particular step in solving the equation, but have difficulty writing the correct result of doing this step, click the *Help* button. The computer will now give the result of doing the step, as in *Balance 1*. A scoring system is used to discourage you from using this help facility too often. The more often you click the *Help* button, the lower your score will be.

Note: the computer will not mark the solution as correct unless the operations and the results at each step are both correct.


Solve the equations, using the "balance strategy".
Use the red buttons to select an operation that will be performed on both sides of the equation.
For each step, fill in the new equation.
You can get help with the help-button, but you get less points.
A correct solution gets you at least 5 points
Without help this is 10 points per problem.
Make an equation yourself
Back
Help
$2(5x - 1) - 4 = -2x + 3(2x - 6)$
distributive law
$10x - 2 - 4 = -2x + 6x - 18$
combine alike terms
$10x - 6 = 4x - 18$
equal with:
$6x = -12$
$\div 6$
$x = -2$
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Scores:
10

A3 Sheet 5 – *Instructions for using the program* Cover up



Purpose

To give practice at solving linear equations with one unknown in which the unknown appears on one side of the equation. This uses the 'cover up' method. Though not as generalisable as the *Balance* method, it is simple to use and helps you to interpret equations. At each step you 'cover up' part of the equation and decide what number must be hidden.

Running the program

The above screen shows how the 'cover up' method may be used to solve the equation:

$$3 + \frac{44}{\sqrt{87 + x}} = 7$$

- 'Cover up' the $\frac{44}{\sqrt{87 + x}}$.

  You can see that the equation takes the form 3 + ... = 7.
  The missing number must therefore be 4.

  Drag the cursor over the term $\frac{44}{\sqrt{87 + x}}$ and it appears on the line below. Type 4 in the box on the right then hit 'Enter'.
  A green (or red) symbol indicates that the step is correct (or incorrect). If you make an incorrect step, click the red button again.

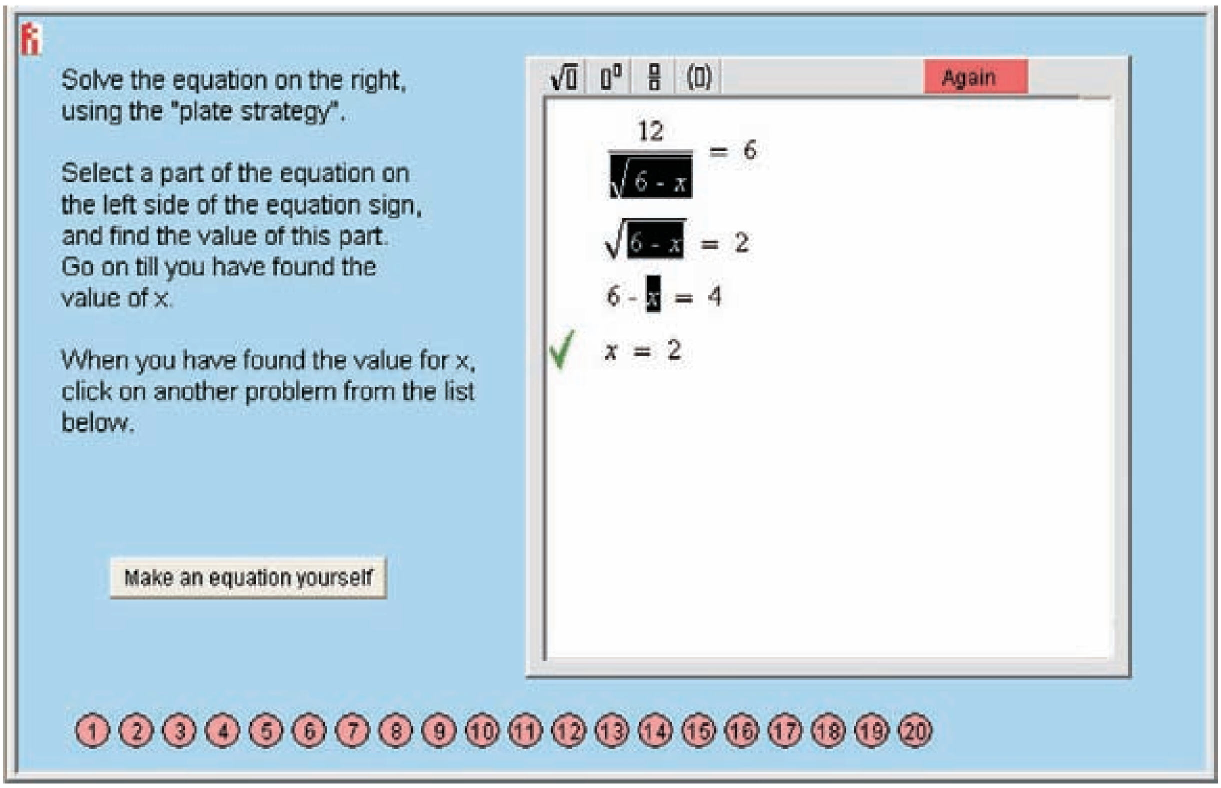
A3 Sheet 5 – *Instructions for using the program* Cover up (continued)

- Cover up the $\sqrt{87 + x}$.
  Now you have an equation of the form 44 ÷ .... = 4.
  The missing number must be 11.
  Drag the cursor over the $\sqrt{87 + x}$ and it appears on the line below.
  Enter 11 in the box on the right.

And so on...

Along the bottom of the screen there are 20 pink circles indicating that there are twenty equations that you can solve. After you have correctly solved an equation, the colour of the circle changes to green.

If you wish, you can make an equation yourself. Click the *Make an equation yourself* button, then enter your equation in the small drop-down window. Click the red [+] button. Your equation appears in the larger screen, where it can be solved.



A4 • Evaluating algebraic expressions

Mathematical goals

To enable learners to:

- distinguish between and interpret equations, inequations and identities;
- substitute into algebraic statements in order to test their validity in special cases.

Starting points

Learners often use letters in algebra without understanding what they mean. Common misconceptions include believing that:

- a letter can only stand for one particular number;
- different letters must stand for different numbers;
- letters can only stand for whole numbers.

Such misconceptions often arise when learners generalise from a restricted range of examples. This session will build on their knowledge of substitution to reconsider such interpretations.

Materials required

For each small group of learners you will need:

- Card set A – *Always, sometimes or never true?*;
- large sheet of paper for making a poster;
- glue stick;
- felt tip pen.

Time needed

About 1 hour.

Suggested approach

Beginning the session

Write the following statement on the board: $x + y = xy$

Ask learners to explain to you what this statement says in words and whether they think it is a true statement or not.

Typically, learners will begin by saying that this is clearly not true because 'add two numbers' does not mean the same as 'multiply two numbers'.

Ask learners for values of x and y which will prove that the statement is not always true. This is not usually difficult.

Now ask if learners can spot any values that make the statement true. Typically, they quickly spot $x = 2$, $y = 2$ and $x = y = 0$. Some learners may reject the case $x = 2$, $y = 2$ because "x and y are different letters so they cannot take the same value". This misconception needs to be discussed explicitly.

The statement is therefore sometimes true. If you wish to challenge learners further, you could ask them to see if they can think of any more cases for which the statement is true, e.g. when $x = 1.5$ or $\frac{3}{2}$, $y = 3$; $x = \frac{4}{3}$, $y = 4$; $x = \frac{5}{4}$, $y = 5$. This makes the point that the examples do not have to be integers.

Explain that in this session learners will be asked to consider a collection of statements in a similar way. The objective of the session is for each group of learners to produce a poster which shows each statement classified according to whether it is always, sometimes, or never true and furthermore:

- if it is sometimes true, then write examples around the statement to show when it is true and when it is not true;
- if it is always true, then to give a variety of examples demonstrating that it is true, using large numbers, decimals, fractions and negative numbers if possible;
- if it is never true, then to say how we can be sure of this.

Working in groups

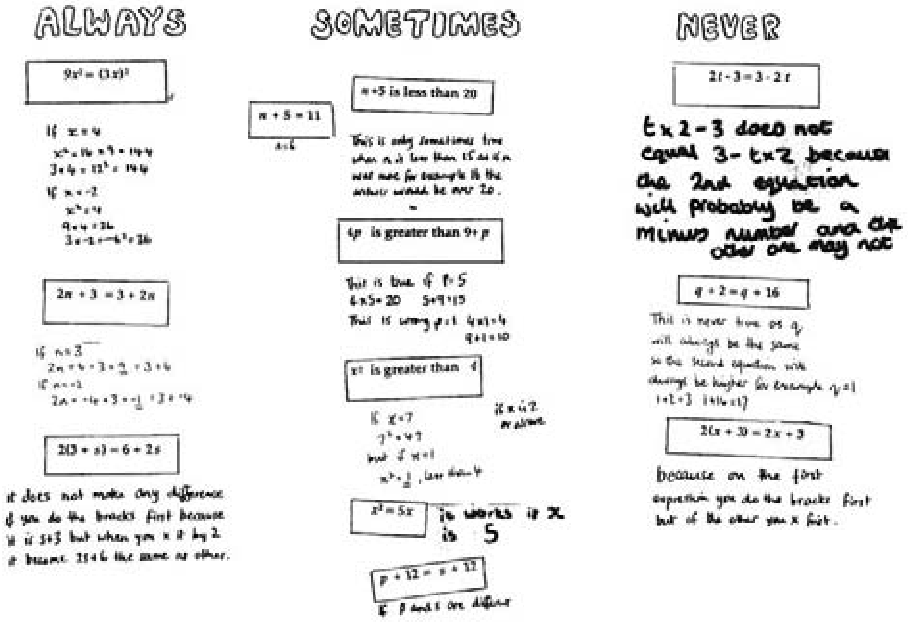
Ask learners to work in pairs or groups of three.

Give each group Card set A – *Always, sometimes or never true?,* a large sheet of paper and a glue stick.

Ask learners to divide their poster into three columns and head these with the words 'Always true', 'Sometimes true', 'Never true'.

Learners should now take it in turns to place a card in one of the columns and justify their answer to their partner(s). Their partner(s) must challenge them if the explanation has not been clear and complete. When the group is in agreement, they should stick the card down and write examples around it to justify their choice. This should include examples and counterexamples.

Learners should not need to rearrange the equations. Trial and error substitutions should prove adequate in most cases (although some may enjoy showing that statements which are always true reduce to $0 = 0$).



When doing this activity, we often find that learners sort their cards quickly and superficially, at least to begin with. They often need prompting to try decimal and negative substitutions to check their assumptions. Learners may change their minds many times; arrows often appear all over the poster showing that a statement was initially classified incorrectly.

Many common beliefs and misinterpretations will surface. Some, for example, appear to believe that different letters have to stand for different numbers and classify $p + 12 = s + 12$ as 'never true'. Some combine letters and numbers inappropriately, thus believing that $3 + 2y = 5y$ is 'always true'. Some appear to believe that $x^2 > x$ is always true because multiplication always makes numbers bigger, and so on. These misconceptions should be picked up in the whole group discussion at the end.

Encourage learners to justify why some expressions are always true (and are therefore identities) using area diagrams (see **A1 Interpreting algebraic expressions**).

On the positive side, you may also notice learners beginning to interpret and reason, using the symbols confidently and meaningfully. In one session, we saw one small group immediately classify $2t - 3 = 3 - 2t$ as 'never true' because "one side is always the negative of the other side". We asked them to reflect on whether changing the sign of a number will always change its value and later returned to find that they had realised that the statement was true when both sides of the equation are zero. That led them to the solution $t = 1.5$.

Reviewing and extending learning

Conclude the session with whole-group questioning using mini-whiteboards.

Ask learners whether they believe that statements such as the following are always, sometimes, or never true, and then ask them to justify their answers. They should hold up their mini-whiteboard showing A, S or N, and give their reasons when challenged.

$x + 2 = 3$

$n + 12 = n + 30$

$x + 6 = y + 6$

$x + y + z = y + z + x$

$3n > n + 3$

$x^2 > 2x$

What learners might do next

Ask learners to create a further set of cards. Their set should contain examples that are as different from one another as possible ($2x + 3 = 5$; $2x + 3 = 6$ are too similar) and should contain an equal number of 'always', 'sometimes' and 'never true' statements.

Learners should produce complete solutions on a separate piece of paper. Then, at a later date, learners can exchange sheets and do each others' questions.

Further ideas

This activity is about examining a series of mathematical statements and deciding on their validity. This idea may be used in many other topics and levels. Examples in this pack include:

SS4 Evaluating statements about length and area;

S2 Evaluating probability statements.

A4 Card set A – *Always, sometimes or never true?*

1 $n + 5 = 11$	2 $q + 2 = q + 16$
3 $2n + 3 = 3 + 2n$	4 $2t - 3 = 3 - 2t$
5 $3 + 2y = 5y$	6 $p + 12 = s + 12$
7 $4p > 9 + p$	8 $n + 5 < 20$
9 $2(x + 3) = 2x + 3$	10 $2(3 + s) = 6 + 2s$
11 $x^2 > 4$	12 $x^2 = 5x$
13 $x^2 > x$	14 $9x^2 = (3x)^2$

A5 • Interpreting distance–time graphs with a computer

Mathematical goals

To enable learners to:

- interpret linear and non-linear distance–time graphs.

Starting points

No prior knowledge is needed.

This session requires the computer program *Traffic* that is supplied with this pack. This program provides a simple yet powerful way of helping learners to visualise distance–time graphs from first principles. The program generates situations involving traffic moving up and down a straight section of road. It then allows the user to take 'photographs' of this situation at one-second intervals, places these side-by-side, and then gradually transforms this sequence of pictures into a distance–time graph. In this way, direct correspondences between speeds and gradients are obtained.

Learners are asked to describe situations, and draw and interpret distance–time graphs. Later, examples are offered that involve cars travelling at non-uniform speeds.

Materials required

An interactive whiteboard or data projector is very useful for demonstrating the computer program and discussing the problems it raises. This is not essential, however.

For each small group of learners you will need:

- a computer loaded with the program *Traffic*.

For each learner you will need one copy of each of the following:

- Sheet 1 – *Traffic situations*;
- Sheet 4 – *Interpreting graphs of traffic situations*;
- Sheet 6 – *The swimming race.*

For each learner you will need several copies of each of the following:

- Sheet 2 – *Blank photographs*;
- Sheet 3 – *Blank graphs*;
- Sheet 5 – *Inventing new situations.*

Time needed

Approximately 1 hour.

Suggested approach

Beginning the session

Give each learner a copy of Sheet 1 – *Traffic situations* and ask them to predict what will happen in each of the situations illustrated. Encourage learners to write their answers in words and to sketch a distance–time graph to show what happens, if they can.

Whole group discussion

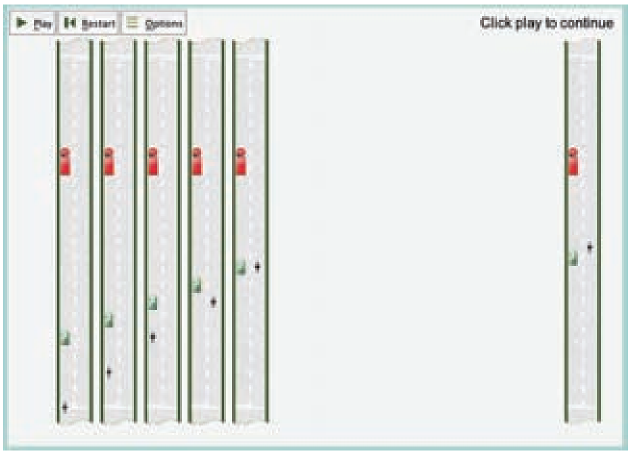
Start the computer program *Traffic* and display it on the interactive whiteboard or data projector, if one is available.

Select from the menu the example 'Velocity 7' and ensure that only the option 'Road' is checked. The computer should now show Situation 1 on Sheet 1. Explain that you now have an aerial view of the road on the screen. Ask learners to read out some of their predictions of what they think will happen to the vehicles and the order in which they think these things will happen.

Click on 'Play' and check their predictions.

Now click 'Options' and tick the buttons marked 'Photos', 'Markers' and 'Graph'. Show the first few photographs of the situation, and click 'Pause'.

> On the left hand side of the screen are photographs of the situation taken at one second intervals. These are laid out side-by-side. Can you predict what the following photos will look like?



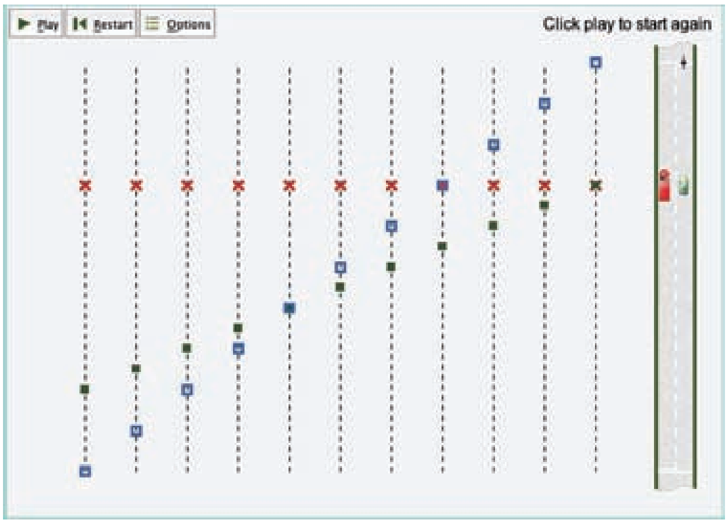
Hand out copies of Sheet 2 – *Blank photographs* and Sheet 3 – *Blank graphs* and ask learners to fill in their second-by-second predictions. Now press 'Play' and ask learners to check whether their predictions were correct.

> From a set of photos, how can you tell that a vehicle is stationary? How can you tell that a vehicle is travelling quickly? How can you tell that it is travelling slowly?

What would the photos look like if we shot two photos per second and laid them out side by side? 20 photos per second?

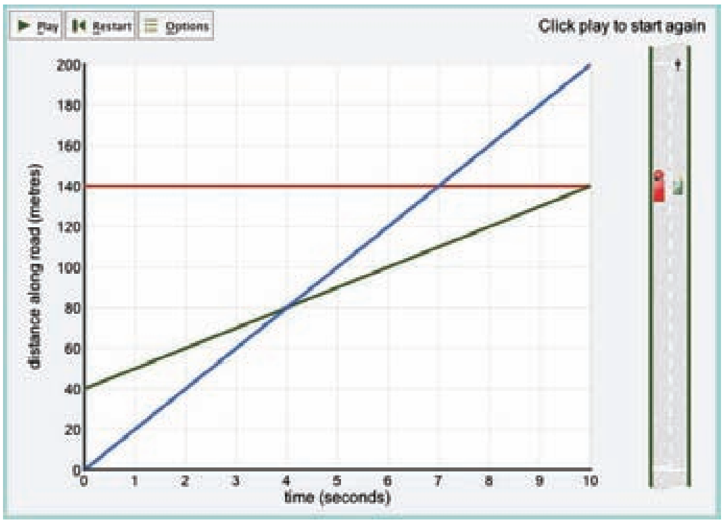
The last question may be used to draw out the continuity of the situation.

Explain that the photo predictions are a bit like a distance–time graph of the situation. In a distance–time graph, we represent the situation at each second using a vertical line, not by a picture of the road. Click on 'Play' to show this transition.



Explain that a graph must also show scales. Click on 'Play' again to show this.

Finally, the graph shows the situation at every instant of time, rather as if we were using a cine-film. Click on 'Play' again to show the distance–time graph.



How can you tell the order of overtaking from the graph alone?

Situation 2 on Sheet 1 requires interpolation. You may wish to repeat the presentation above using this situation. (On the computer, this is shown as 'Velocity 5'.) This time, give each learner a fresh copy of Sheets 2 and 3 and see if they can answer the questions using the graph paper. If they have difficulty, ask them to use the photo blanks first.

Working in groups

Interpreting distance–time graphs on the computer

Ask learners to sit in pairs at a computer and explore the remaining examples in the program. One possible approach is for learners to interpret a number of graphs, then check their interpretations using the animations.

Ask learners to choose 'Options', and switch off everything except the graph. They should then view the graph for each example and try to interpret what the vehicles are doing in that situation, as precisely as possible. That is, they should give descriptions involving distances, times and speeds. This explanation is provided for learners on Sheet 4 – I*nterpreting graphs of traffic situations.*

For example: 'Velocity 2'.

> In the first three seconds, the car travels 100 m at a constant speed of about 33 m per second. From 3 seconds to 6 seconds, the car travels 40 m at a constant speed of about 13 m per second. From 7 seconds to 10 seconds, the car is stationary. We can see that the car moves at a greater velocity for the first three seconds because the slope of the graph is steeper.

After learners have explored these examples, they should compare their descriptions with those of other learners. Ask them to make notes of any differences that emerge. As you move around the room, listen to learners' explanations and note any misconceptions that emerge for discussion in the final whole group session.

Drawing and interpreting distance–time graphs without the computer

Finally, turn off the computers and ask learners to work in pairs. Each learner should, on their own, invent a description of a traffic situation and draw the accompanying graph. Sheet 5 – *Inventing new situations* is provided for this purpose.

When a learner has done this, the graph (or description) should be hidden (by folding it back) and the description (or graph) passed to the second learner to see if they can produce the missing representation. Any mismatches should be discussed and resolved until the learners have reached agreement.

Reviewing and extending learning

Finally, hold a whole group discussion on the situation described in Sheet 6 – *The swimming race*.

Give each learner a copy of Sheet 6 and read it together slowly. If a learner thinks that a mistake has been made, ask them to describe the mistake carefully and how it should be corrected.

What learners might do next

Learners may find **A6 Interpreting distance–time graphs** a useful follow-up to this session. This takes the ideas further and brings in the measurement of acceleration and deceleration.

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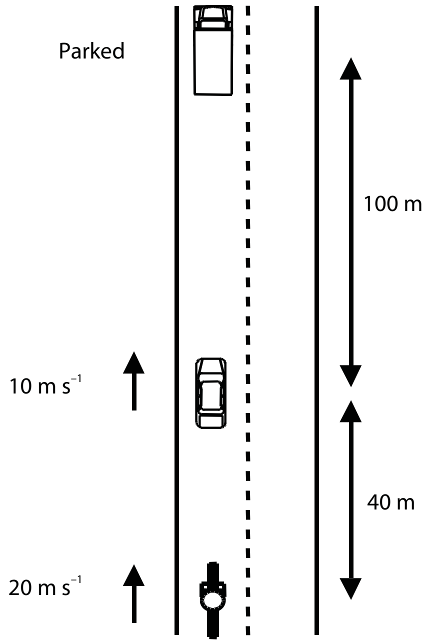
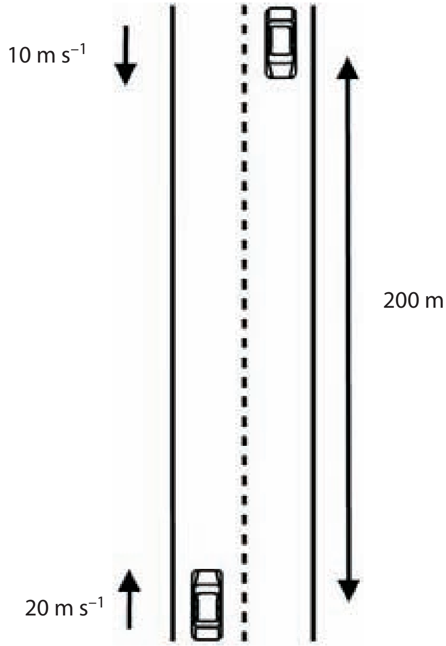
A5 Sheet 1 – *Traffic situations*





Situation 1

This picture shows an aerial view of a narrow country road.

If the vehicles continue to travel at the same steady speeds, what will happen in the next few seconds?

Describe your answers using words, sketches and/or a graph.

Answer

Situation 2

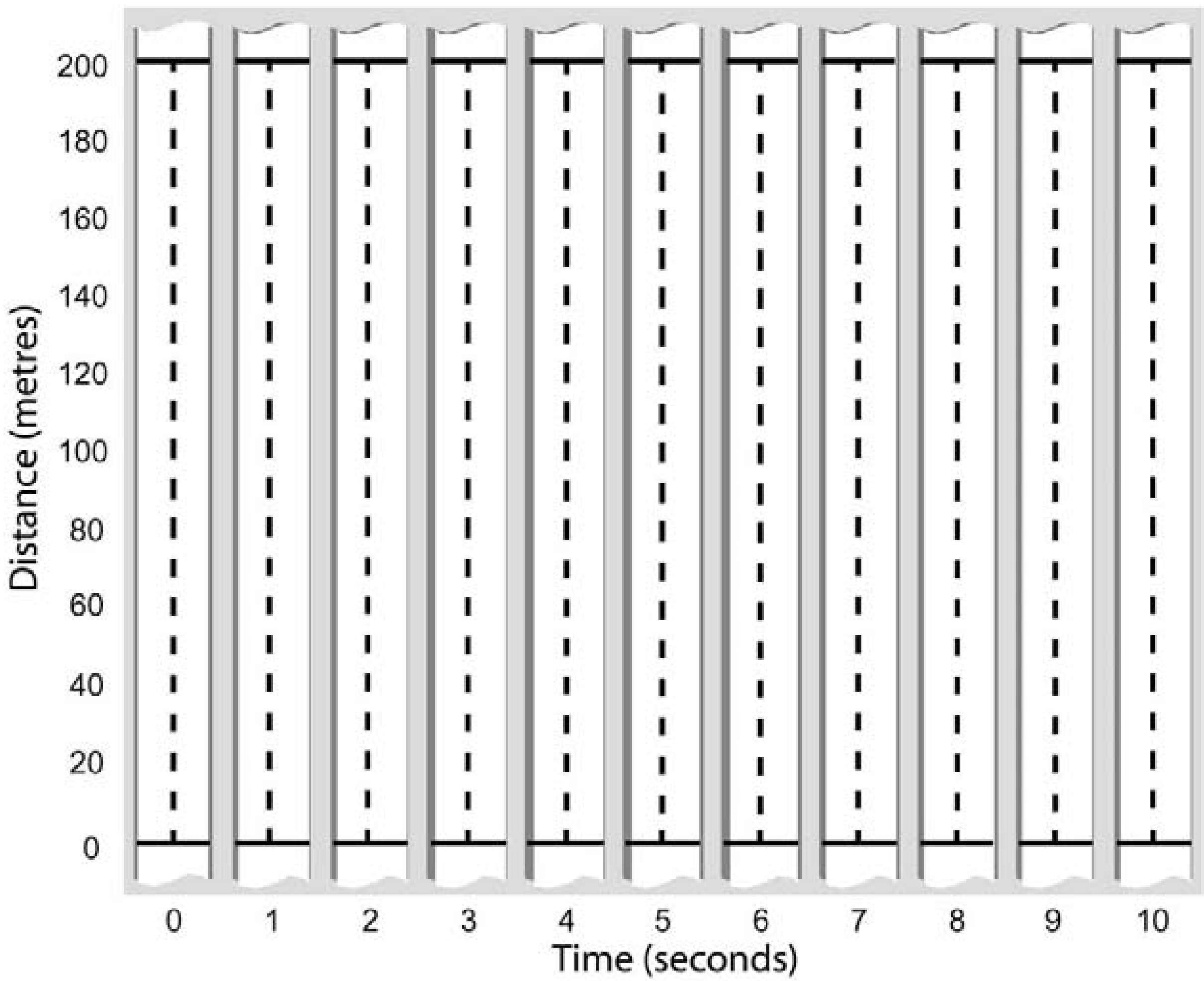
When will these two cars meet?

Where will they be along the road at this time?

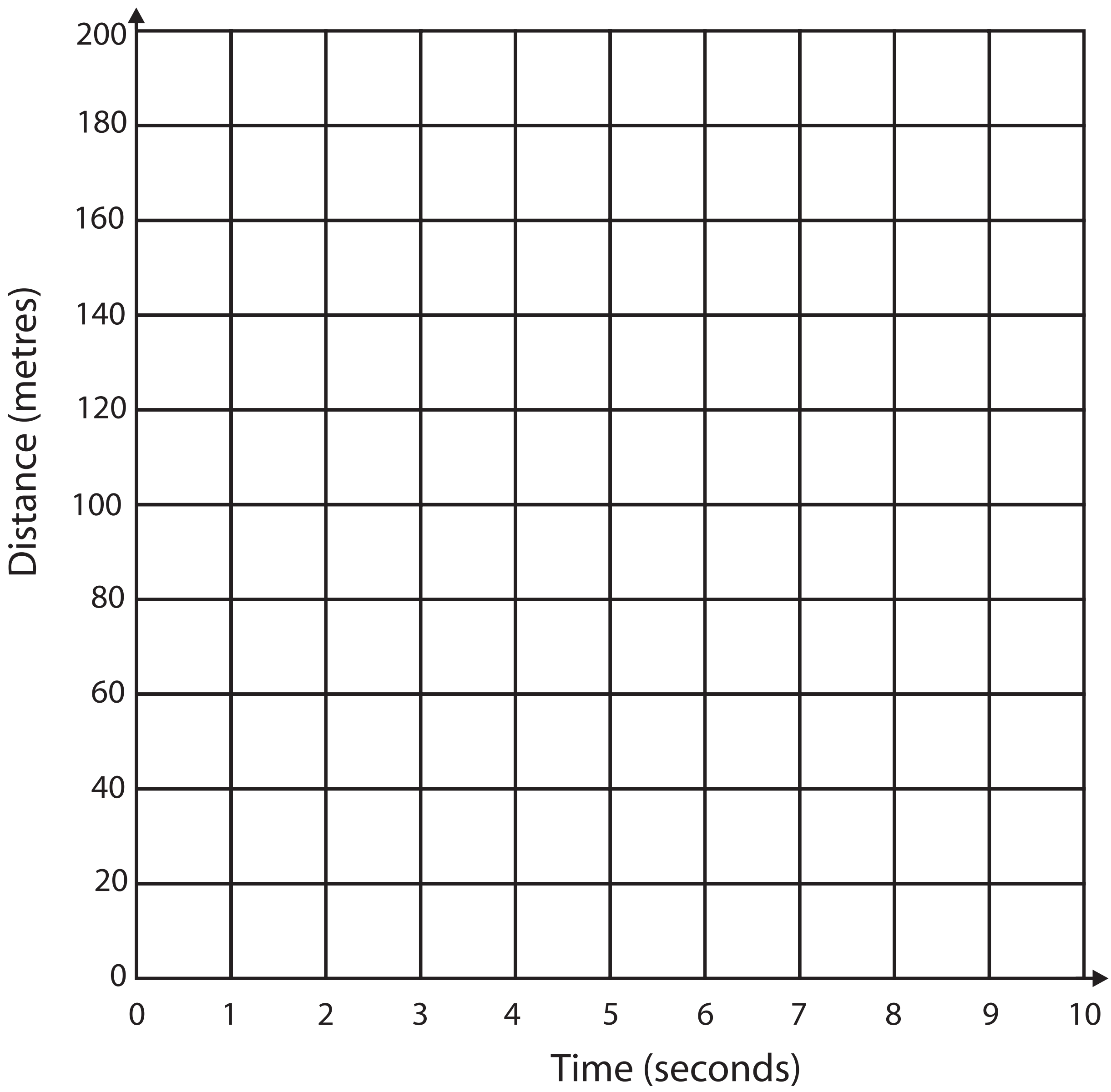
Explain how you know this.

Answer

A5 Sheet 2 – *Blank photographs*

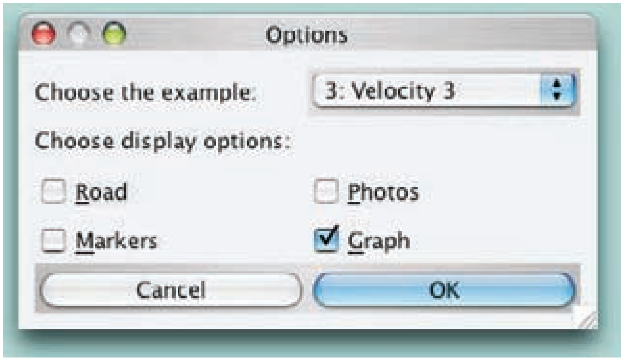


A5 Sheet 3 – *Blank graphs*



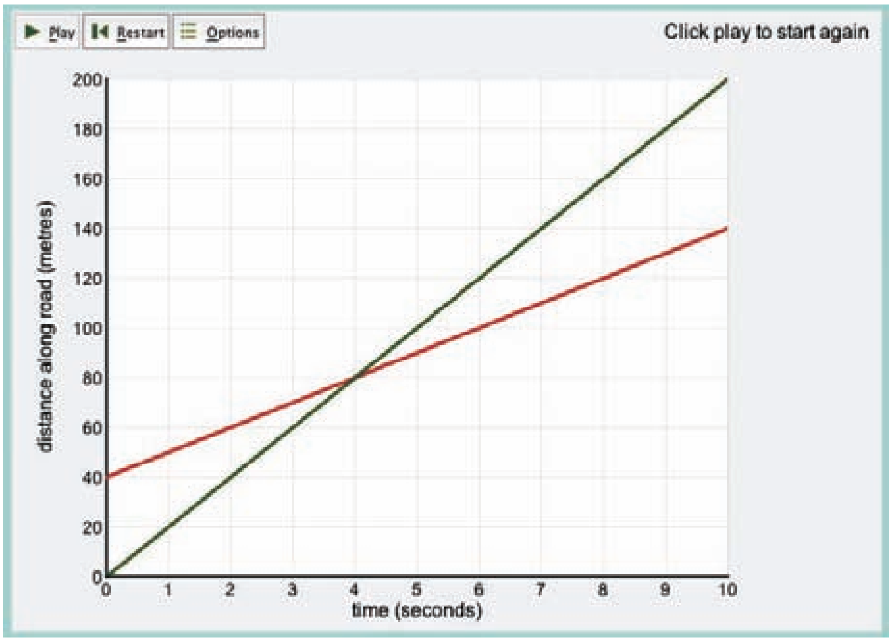
A5 Sheet 4 – *Interpreting graphs of traffic situations*

Choose 'Options' and switch off everything apart from the 'Graph' option.



Now choose each situation in turn. Write a short story saying what you think is happening to the vehicle(s) in that situation. Include details such as speeds.

For practice, complete the details in this example 'Velocity 3'.



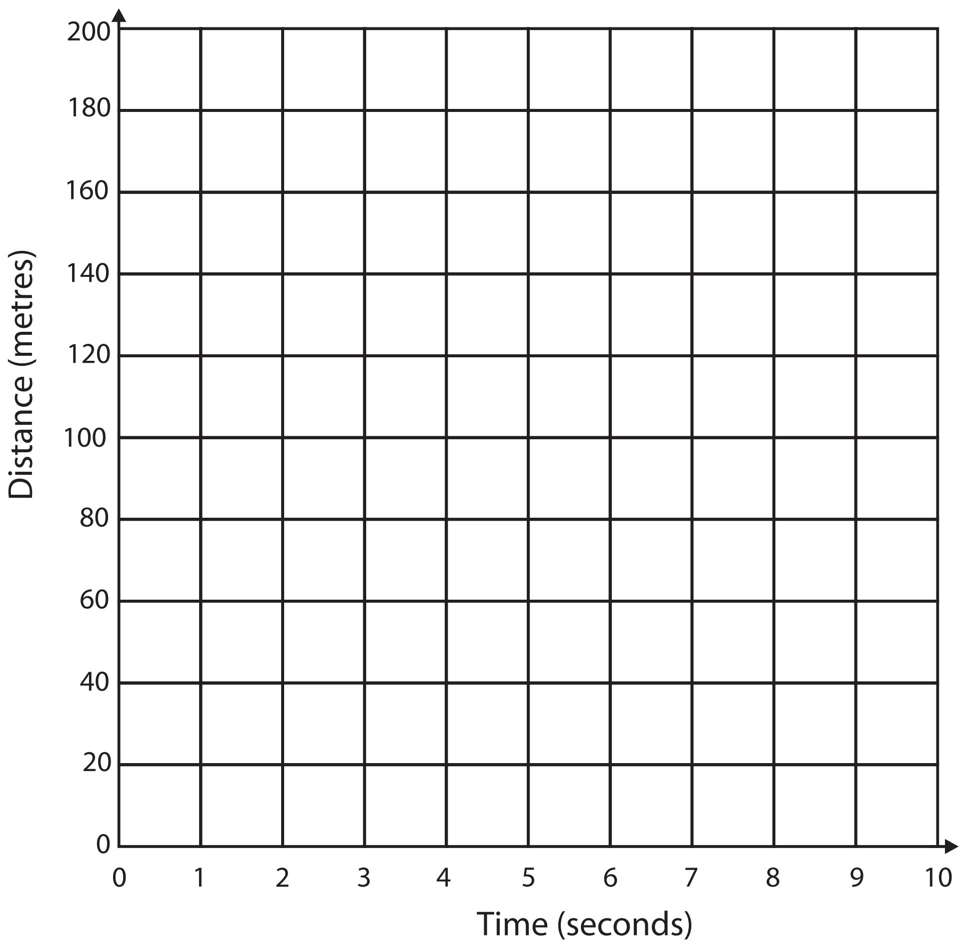
The green car is travelling along at a steady speed of ...... metres per second. The driver of the green car sees a red car 40 metres in front of her. The red car is travelling more slowly at ...... metres per second. After ...... seconds, the green car overtakes the red car.

Now check your story by selecting 'Options' in 'Velocity 3' and selecting 'Road' and 'Graph'. Click 'Play' to see the motion and graph develop together.

A5 Sheet 5 – *Inventing new situations*

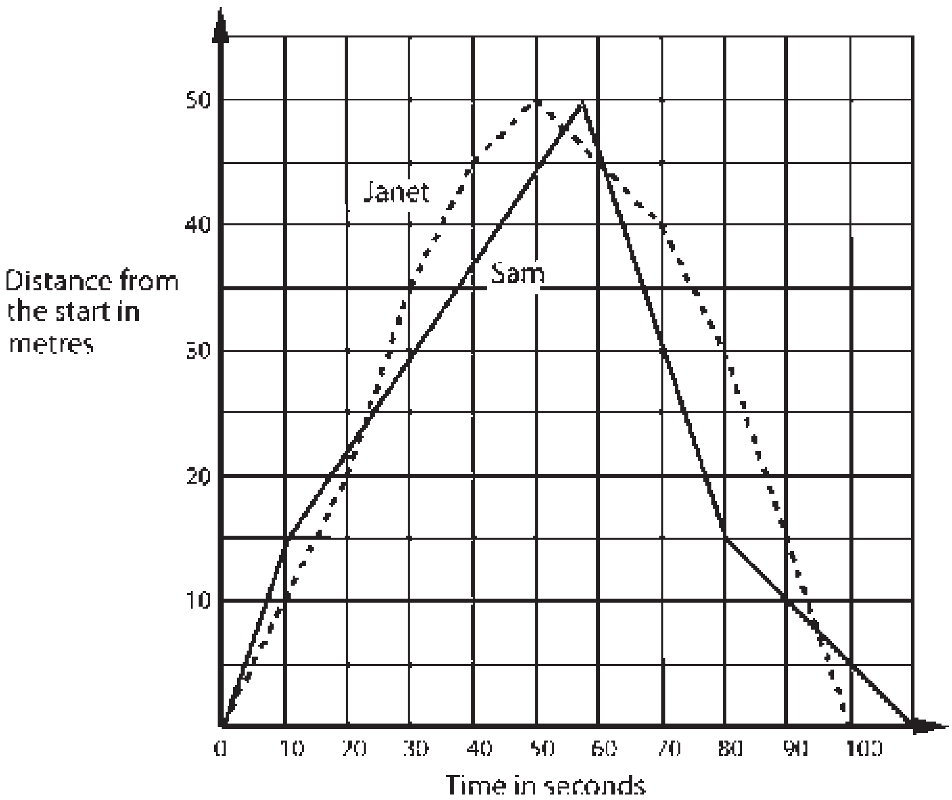
Description

Graph



A5 Sheet 6 – *The swimming race*

The following graph describes a swimming race.



In an exam, a learner was asked to write a commentary to go with this graph. Check her answer and see how many mistakes you can find.

The race commentary:

> Sam goes quickly into the lead. He is swimming at 15 metres per second. Janet is swimming at only 10 metres per second. After 22 seconds, Janet overtakes Sam. Janet swims more quickly than Sam from 25 seconds until she turns at 50 seconds. Sam overtakes Janet after 55 seconds, but she catches up again, 5 seconds later. Janet is in the lead until right near the end. Sam swims at a steady 30 metres per second after the turn, until 80 seconds, while Janet is gradually slowing down. Sam wins by 10 seconds.

Now try to write a better commentary.

A6 • Interpreting distance–time graphs

Mathematical goals

To enable learners to:

- interpret and construct distance–time graphs, including:
  - relating speeds to gradients of these graphs;
  - relating accelerations to changes in these speeds.

Starting points

Learners have often constructed distance–time graphs before. However, experience shows that many still interpret them as if they are pictures of situations rather than abstract representations. In addition, they also find it difficult to interpret the significance of the gradients of these graphs.

In this session, learners begin by discussing a question that is designed to reveal common misconceptions about distance–time graphs. They then work in pairs and threes to match descriptions, graphs and tables. As they do this, they will interpret their meaning and begin to link the representations together.

Materials required

- OHT 1 – *Speed and acceleration: car*;
- OHT 2 – *Speed and acceleration: motorbike*.

For each learner you will need:

- Sheet 1 – *Journey to college; Motorbike and car*;
- Sheet 2 – *The race*;
- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Descriptions*;
- Card set B – *Distance–time graphs*;
- Card set C – *Distance–time tables*;
- Card set D – *Speeds and accelerations*.

Time needed

Approximately 1 hour 30 minutes.

Suggested approach

Beginning the session

Ask learners to tackle the two questions on Sheet 1 – *Journey to college* and *Motorbike and car*, working in pairs. Both questions ask learners to interpret distance–time graphs.

As you listen to pairs tackling the first question, watch for evidence of misconceptions such as the following:

> When she gets out she starts walking fast to the bus stop, then she slows down, then she picks up speed again, and then the speed goes constant.
>
> Jane walked along a road for 100 m. Instead of walking another 30 m she took a short cut down an alleyway which took her 20 minutes. She walked very quickly then she caught the bus to her college which took about 50 minutes.

In the second question, many learners assume that the vehicles are travelling at the same speed at the point where the graphs cross.

Encourage learners to discuss their answers with other pairs. Do not comment at this stage. After a few have shared their ideas, begin to challenge them for more details, such as the speed at each point on the graphs. Ask learners to describe the evidence for each aspect of their description. If the errors in the two stories quoted above do not arise naturally, read the stories out and ask learners to explain the mistakes.

Finally, use OHTs 1 – *Speed and acceleration: car* and 2 – *Speed and acceleration: motorbike* to describe how the speeds and accelerations are related to the gradients of the graphs.

Working in groups

Ask learners to work in groups of two or three and give out Card sets A – *Descriptions* and B – *Distance–time graphs*.

Ask learners to take it in turns to match pairs of cards from each set. This is not a one-to-one matching. If learners think that any cards are missing, they should create their own.

When learners have had enough time to tackle the task, give out Card sets C – *Distance–time tables* and D – *Speeds and accelerations*. These should be matched to the cards already on the table. Although there are no scales on the graphs on Card set D, learners may be able to work out the correct matching by considering the differences between successive terms in the tables.

Reviewing and extending learning

Ask learners some 'show me' questions using mini-whiteboards:

Show me a distance–time graph for:

- a car travelling at a steady speed;
- a car speeding up;
- a car slowing down;
- a stationary car;
- two cars travelling at the same speed towards each other;
- a car is crawling along in the slow lane and a car overtakes very quickly;
- a child runs into the road, so the driver has to make an emergency stop;
- a car slows down as it goes over a speed bump, then goes quickly again.

Finally, encourage learners to tackle the problem in Sheet 2 – *The race*. This will help to consolidate what they have learned during the session.

What learners might do next

Invite learners to create their own distance–time graph and a story that matches it.

Ask learners to exchange stories and try to recreate the graphs from the stories alone. Finally, compare the resulting graphs with the originals and discuss the discrepancies.

Further ideas

This activity uses multiple representations to deepen understanding of distance–time graphs. This type of activity may be used in any topic where a range of representations is used. In this pack, other examples include:

N5 Understanding the laws of arithmetic;

A1 Interpreting algebraic expressions.

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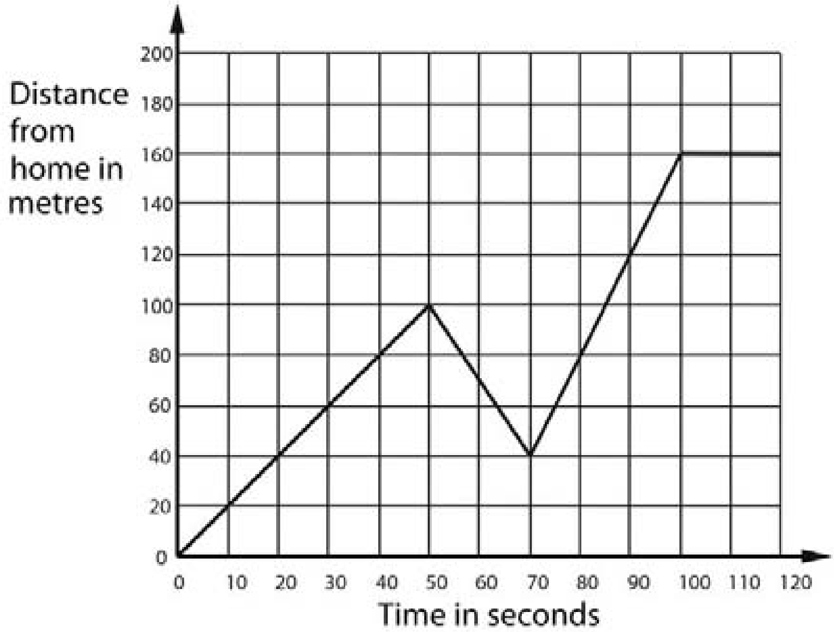
A6 Sheet 1 – *Journey to college; Motorbike and car*

Journey to college



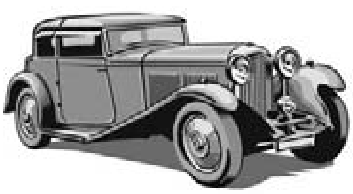
Every morning Jane walks along a straight road to a bus stop 160 metres from her home, where she catches a bus to college.

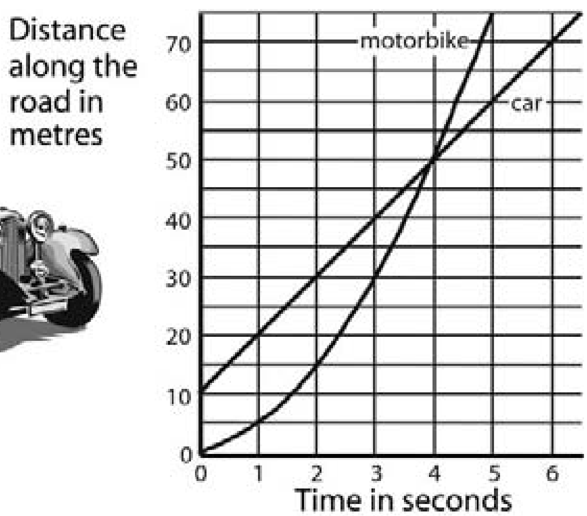
The graph shows her journey on one particular day.

Write a description of what may have happened.

You should include details such as how fast she walked.

Motorbike and car

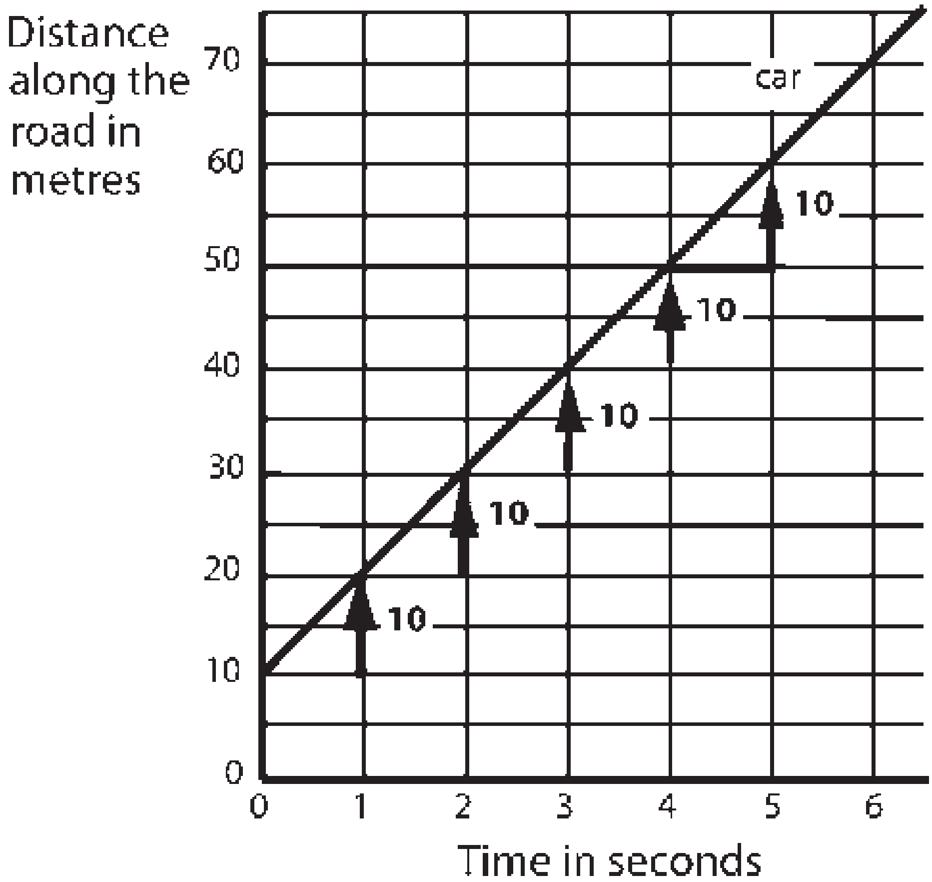




This graph shows a vintage car and a motorbike travelling along a country road.

Write a description of what is happening. At what time are they travelling at the same speed?

A6 OHT 1 – *Speed and acceleration: car*

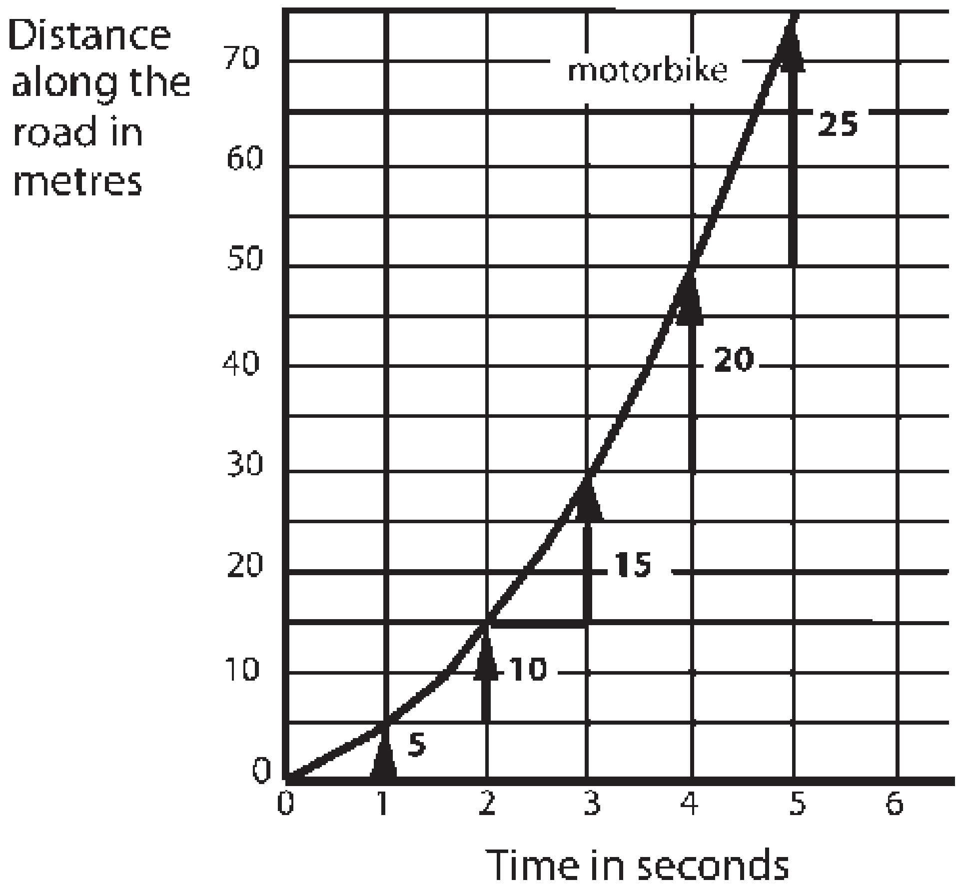


Time (s)	0	1	2	3	4	5	6
Distance (m)	10	20	30	40	50	60	70

→ +10 → +10 → +10 → +10 → +10 → +10

Car travels at a constant speed of 10 m s^{-1}.

A6 OHT 2 – *Speed and acceleration: motorbike*



Time (s)	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Distance (m)	**0**	**5**	**15**	**30**	**50**	**75**	**?**

→ +5 → +10 → +15 → +20 → +25 → +?

Each second, the average speed of the bike increases by 5 m s^{-1}.

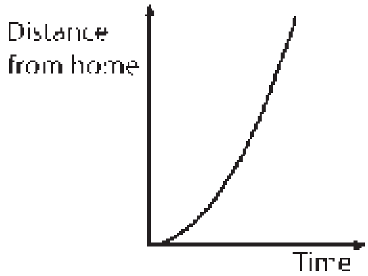
Acceleration is 5 m s^{-2}.

A6 Card set A – *Descriptions*

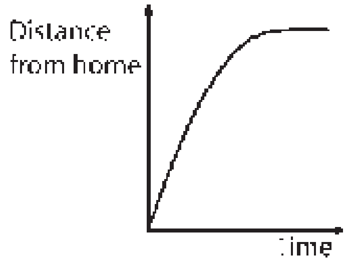
Travels towards home	Travels away from home
Goes at a steady, slow speed	Not moving
Impossible journey	Goes at a steady, fast speed
Stops suddenly	Slows down and stops
Speeds up	

A6 Card set B – *Distance–time graphs*

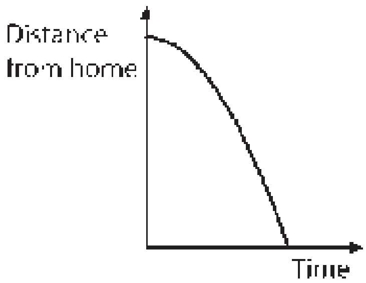
A



B



C



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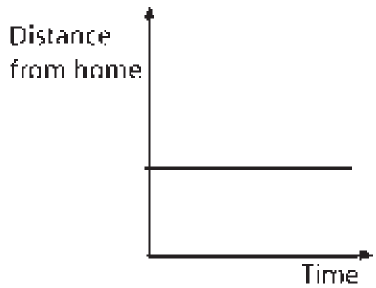
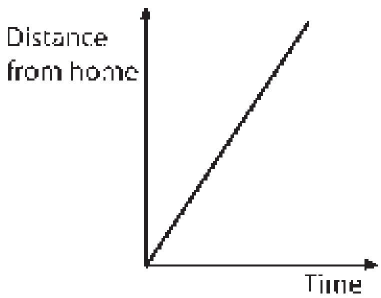


E



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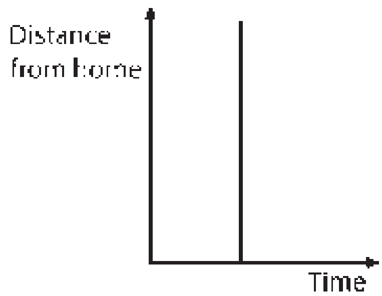
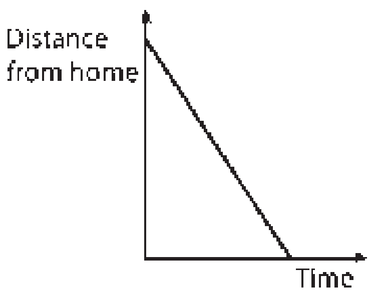
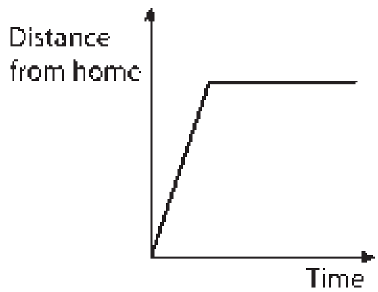


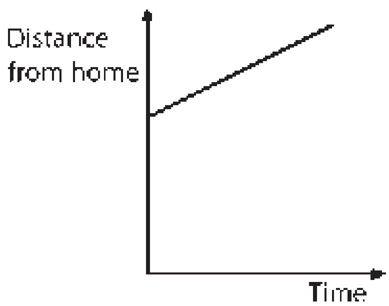
G



H



I



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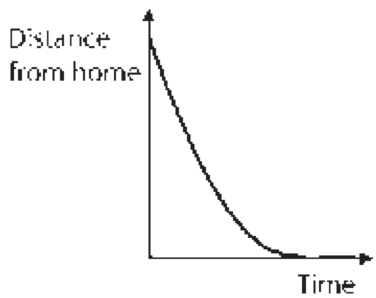


A6 Card set C – *Distance–time tables*

A

Time	0	1	2	3	4	5
Distance	0	45	80	105	120	125

B

Time	0	1	2	3	4	5
Distance	125	120	105	80	45	0

C

Time	0	1	2	3	4	5
Distance	0	5	20	45	80	125

D

Time	0	1	2	3	4	5
Distance	125	80	45	20	5	0

E

Time	0	1	2	3	4	5
Distance	50	50	50	50	50	50

F

Time	0	1	2	3	4	5
Distance	0	25	50	75	100	125

G

Time	0	1	2	3	4	5
Distance	125	117	109	101	93	85

A6 Card set C – *Distance–time tables*

H

Time	0	1	2	3	4	5
Distance	85	93	101	109	117	125

I

Time	0	1	2	3	4	5
Distance	125	100	75	50	25	0

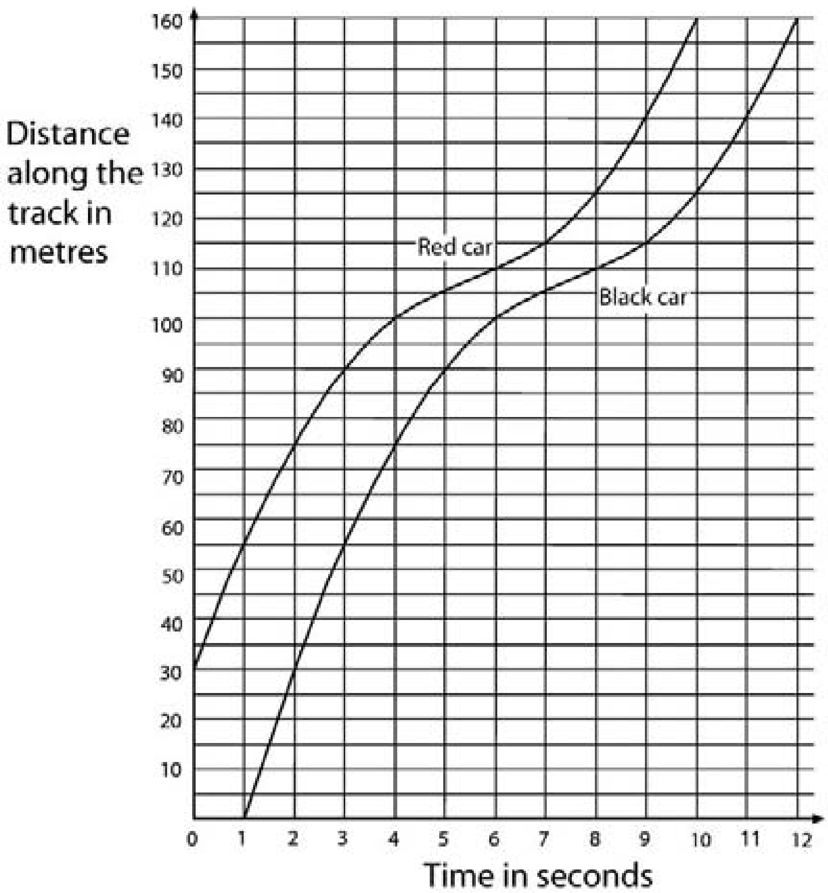
J

Time	0	1	2	3	4	5
Distance						

A6 Card set D – *Speeds and accelerations*

Speed **25 m s^{-1}**	**Acceleration** **10 m s^{-2}**
Speed **8 m s^{-1}**	**Deceleration** **10 m s^{-2}**
Speed **0 m s^{-1}**	**Acceleration** **0 m s^{-2}**

A6 Sheet 2 – *The race*



The graph shows the motion of two racing cars as they approach and go round a bend on a racetrack.

1. How far along the track is the bend?
2. Make tables to show how the distance travelled by each car changes with time.
3. How does the distance between the two cars vary?
4. How does the time interval between the two cars vary?
5. Can you work out the deceleration and acceleration of each car?
6. During motor racing events, it seems that the car in front loses some of its lead when it approaches a bend, and then opens up a gap again afterwards. Why is this? Are the chasing cars really catching up?

A7 • Interpreting functions, graphs and tables

Mathematical goals

To enable learners to understand:

- the relationship between graphical, algebraic and tabular representations of functions;
- the nature of proportional, linear, quadratic and inverse functions;
- doubling and squaring, and the effect on positive and negative numbers.

Starting points

Learners should already be familiar with algebraic symbols such as those representing squares, square roots and fractions. This will be revised during the introduction. Most learners will already be familiar with making tabular representations of functions and drawing graphs, but they may not have considered the relations between different representations of functions.

Materials required

For each learner you will need:

- calculator;
- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Introductory activity*;
- Card set B – *Formulae*;
- Card set C – *Words*;
- Card set D – *Graphs*;
- Card set E – *Tables*;
- Card set F – *Variations.*

The computer program *Machines* may be used to enrich the work in this session. It is included on the DVD and is also available, with many others, at the Freudenthal Institute website www.fi.uu.nl (on the website it is called 'Algebra arrows', or 'Algebra pijlen'). It is used here with kind permission. Instructions for using this program are given on Sheet 1 – *Instructions for using the software: machines.*

Time needed

At least 1 hour.

Suggested approach

Beginning the session

The first part of the session aims to help learners recall the meaning of algebraic representations.

Arrange learners into pairs. Give each pair Card set A – *Introductory activity*.

Ask them to imagine that $n = 4$ and then try to place the cards in order of size. If they struggle, ask them to say in their own words what each symbol means. When they have placed the cards in order, ask them to repeat the task when n takes other values, for example $n = 9$, then $n = \frac{1}{4}$. Allow the use of calculators for this activity.

To enliven the activity, invite six learners out to the front and give each one a large version of a card. They must then place themselves in order, using suggested substitutions from the rest of the group. To make things more difficult, you could ask the group to nominate one standing learner and then find a substitution that will make this person stand at one end of the line.

Working in groups

Hand out Card sets B – *Formulae* and C – *Words* to each pair of learners. Ask learners to take it in turns to try to match a pair of cards and to explain why these two cards are equivalent. This process will help learners to articulate the meaning of the cards. Where there are blank cards, learners should complete these themselves.

It is helpful if learners are asked to place the cards side by side so that you can monitor their work as you move round the room. Some of the cards are more difficult in that they require the formula to be slightly rearranged.

When they have completed this, give learners Card sets D – *Graphs* and E – *Tables,* to be matched with those already on the table. These cards encourage learners to substitute numbers into each algebraic representation and to link it with a graphical representation.

If the table becomes too cluttered at this point, you may suggest that learners remove the *Words* cards.

If learners find the work difficult, suggest that they work with the easier cards first: say, $y = 2x$, $y = x + 2$, $y = 2$, $y = x - 2$, $y = \frac{x}{2}$.

Once they have correctly matched these cards, they may be ready to match the remaining cards.

Learners who complete the activity quickly may enjoy trying to match Card set F – *Variations* with the sets on the table. These invite learners to consider the effect on one variable as the other is changed.

Reviewing and extending learning

During the final part of the session, use mini-whiteboards and questioning to see if learners can begin to generalise.

Show me a graph of:

- $y = x + 3$;
- $y = -x + 3$;
- $y = \frac{3}{x}$.

Show me:

- an equation of a straight line graph that goes through the origin;
- . . . and another; and another; now a steeper line;
- an equation of a straight line graph that goes through (0,2);
- . . . and another; and another; now a steeper line;
- an equation of a line with a negative gradient;
- an equation of a parabola that goes through (0,3);
- and so on . . .

What learners might do next

Learners may use the computer program *Machines* to construct function machines and explore their graphs. Instructions are given on Sheet 1 – *Instructions for using the software: machines*.

Further ideas

This activity uses multiple representations to deepen understanding of functions. This type of activity may be used in any topic where a range of representations is used. Examples in this pack include:

A6 Interpreting distance–time graphs;

SS6 Representing 3D shapes.

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A7 Card set A – *Introductory activity*

n	n^2
$\sqrt{n}$	$8n$
$\dfrac{36}{n}$	$\dfrac{n}{2} + 1$

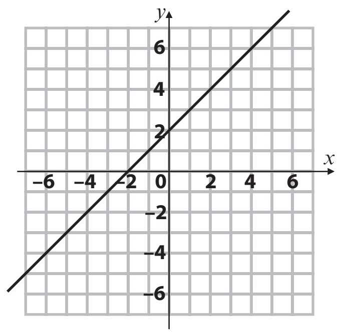
A7 Card set B – *Formulae*

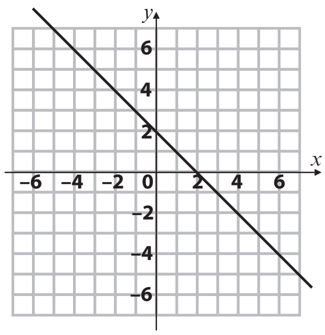
$y = x^2$	$y = 2x$
$y^2 = x$	$y = x + 2$
$2y = x$	$y = 2$
$y = x - 2$	$xy = 2$
$y = \pm\sqrt{x}$	$x + y = 2$
$y = \frac{2}{x}$	$y = \frac{x}{2}$
$y = -x + 2$	$x = \pm\sqrt{y}$

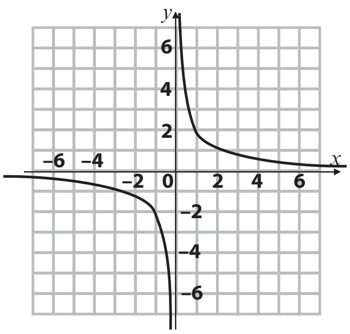
A7 Card set C – *Words*

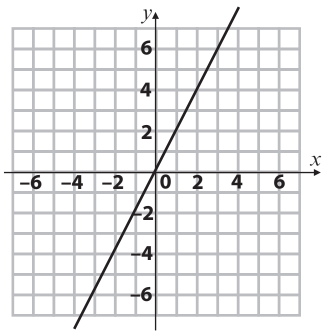
y is one half the size of x	x added to y is equal to 2
y is 2 more than x	x multiplied by y is equal to 2
y is 2 less than x	y is double the size of x
y is always equal to 2	x is the same as y multiplied by y
y is the same as 2 divided by x	y is the same as x multiplied by x
x is the square root of y	y is the same as x divided by 2

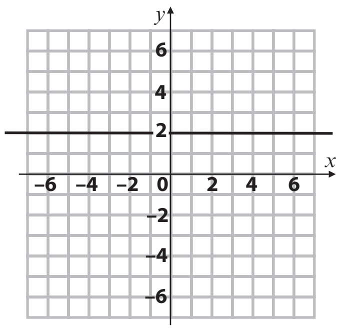
A7 Card set D – *Graphs*

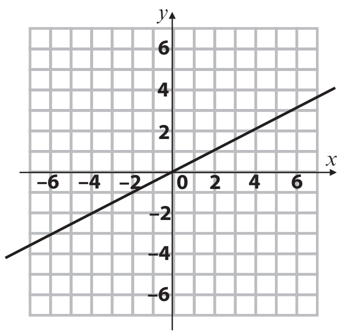


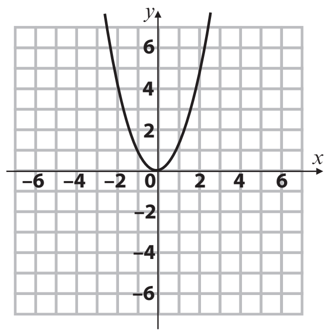


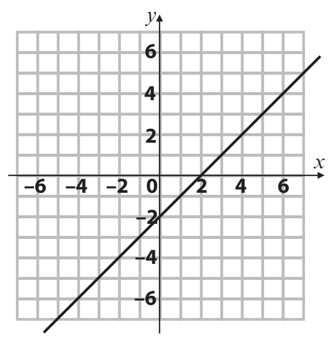


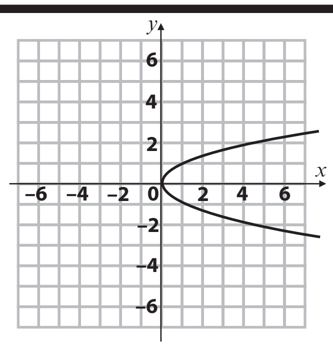












A7 Card set E – *Tables*

x	–2	–1	0	1	2	3
y	–4	–3	–2	–1	0	1

x	–2	–1	0	1	2	3
y	4	3	2	1	0	–1

x	–2	–1	0	1	2	3
y	–1	–0.5	0	0.5	1	1.5

x	–2	–1	0	1	2	3
y	–4	–2	0	2	4	6

x	0	1	4	9	16
y	0	±1	±2	±3	±4

x	–2	–1	0	1	2	3
y	0	1	2	3	4	5

x	–2	–1	0	1	2	3
y	4	1	0	1	4	9

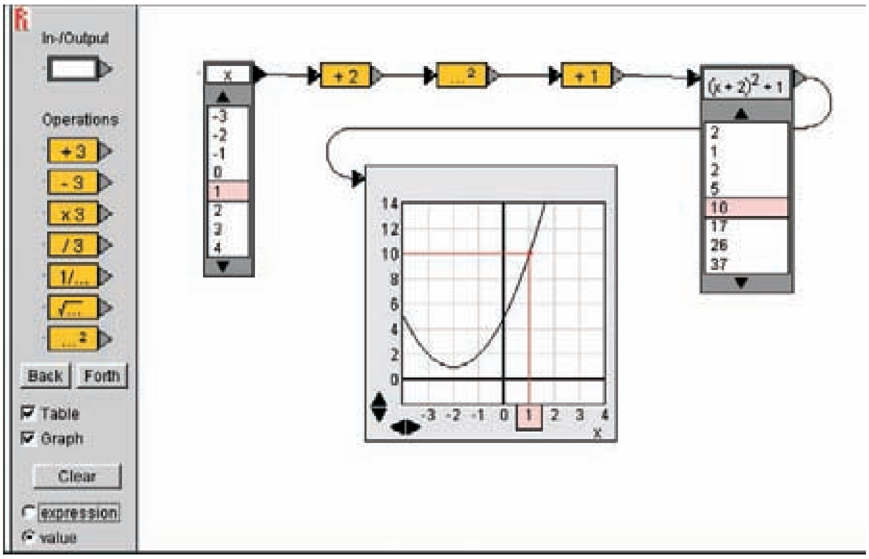
x	–2	–1	0	1	2	4
y	–1	–2	±∞	2	1	0.5

x	–2	–1	0	1	2	3
y	2	2	2	2	2	2

A7 Card set F – *Variations*

If you double x, y doubles.	**If you double x, y halves.**
If you add 1 to x, 2 is added to y.	**If you add 1 to x, 1 is added to y.**
If you add 1 to x, one half is added to y.	**If you double x, y doesn't change.**
If you double x, y is multiplied by 4.	**If you multiply x by 4, y is multiplied by 2.**

A7 Sheet 1 – *Instructions for using the software: machines*



This program provides an interactive way of creating function machines with one instance of the variable. It allows learners to explore connections between the machines, algebraic formulae, tables of numbers and graphs.

Running the program

As an example, suppose you want to make a machine for the function $y = (x + 2)^2 + 1$.

- Drag an input box onto the white screen area.
- The operations you want to do, in order are: +2, square, +1. At the moment, these are not all shown on the left hand side, so you must make them. Drag a yellow +3 box into the white screen. Now click on the +3 and change it to +2. Drag the arrow from the right hand side of the input box to the +2 yellow box. It should attach itself.
- Drag the yellow function box for squaring [....^{2}] onto the white screen and attach an arrow from the +2 function to this box. Finally repeat the process and create the +1 function.
- Now join this onto an output box. You now have finished making the machine.
- If you want to remove anything from the screen, just drag it off.

Try typing a number into the input box. Do you get the number you expect in the output box?

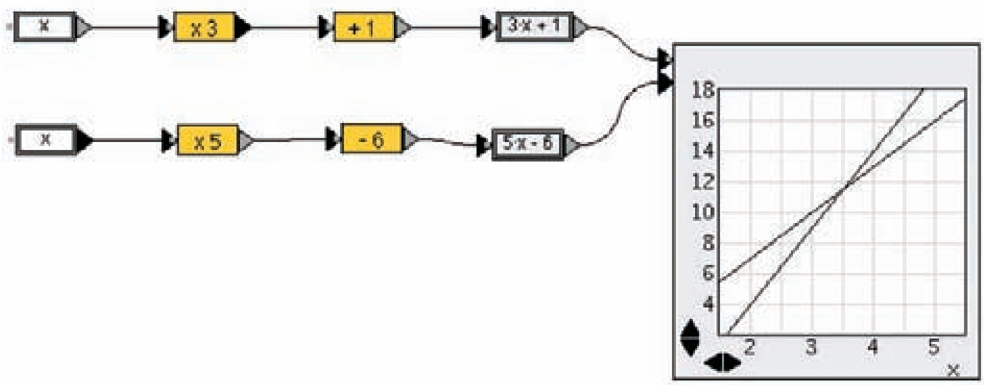
Try typing x into the input box. Do you get the correct formula in the output box?

With x in the input box, try clicking on the 'Table' button. What happens? Click on a number in the 'Table' box and see what happens now. You can scroll up and down the table boxes by clicking on the arrows at the top and bottom.

Click on the 'Graph' button and join the arrow from the output box onto the graph. Drag the inside of the graph around and adjust the scales by clicking on the arrows until you can see it clearly.

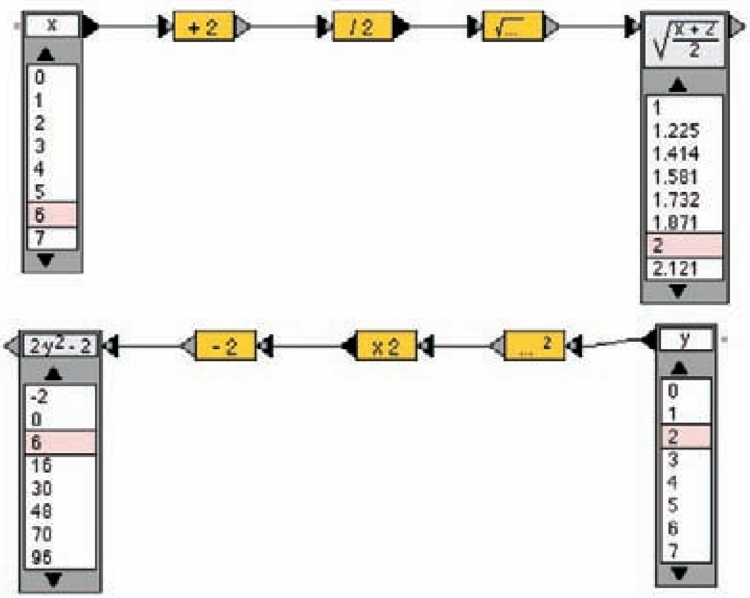
A7 Sheet 1 – *Instructions for using the software: machines* (continued)

You can of course have more than one function machine on the graph at the same time. This is useful when solving simultaneous equations. For example, you can solve $y = 3x + 1$ and $y = 5x - 6$ graphically:



If you click on the 'Back' button, the operations are all reversed. This is useful when showing inverse functions.

For example, the inverse function for $y = \sqrt{\dfrac{x+2}{2}}$ is $x = 2y^2 - 2$



A8 • Developing an exam question: generalising patterns

Mathematical goals

To help learners to:

- use past examination papers creatively;
- explore, identify, and use pattern and symmetry in algebraic contexts, investigating whether a particular case can be generalised further;
- understand the importance of counter-examples;
- develop the ability to generalise from geometric patterns;
- devise and explore their own questions in this context.

Starting points

Most learners will be familiar with looking for growth patterns in geometric patterns and many will be able to express these patterns verbally. However, it may be that they will find it more difficult to express the nth term algebraically.

Materials required

For each small group of learners you will need:

- Sheet 1 – *Growing cross patterns*;
- Sheet 2 – *Template for growth patterns*;
- a supply of squared or triangular dotted or lined paper.

Time needed

About 1 hour.

Suggested approach

Beginning the session

Ask learners to work in pairs on the GCSE examination question in Sheet 1 – *Growing cross patterns*. When everyone has had time to have a go at this, ask them to gather round for a whole group discussion on the approaches used.

Whole group discussion (1)

(i) Answering the question

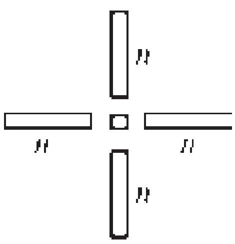
Ask learners to describe their methods for tackling part (a) of the question.

> Chris, why do you say that there are 21 squares in diagram 6?
> (It goes up in fours: 5, 9, 13, 17, 21.)
>
> Jack, tell me why you thought the answer was 26 squares.
> (You just double the number of squares that are in diagram 3.)
>
> What did you do, Sam?
> (I drew the diagram for six and then counted the squares.)

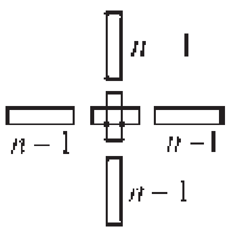
Ask learners to explain the advantages and disadvantages of each method.

> Chris's method works because there are four arms to the diagram and each time you add a square on at the end of each arm.
>
> Jack's method doesn't work. If you double the number for the first diagram you don't get the number for the second diagram.
>
> Sam's method will work, but it takes a long time.

Similarly, invite learners to suggest answers to parts (b) and (c) of the question and to explain their reasoning. Where several different methods are suggested, ask learners to say why they are correct (or not) and to justify why different answers may be equivalent.



> You have one square in the middle then each arm is n squares long. So the formula is $1 + 4n$.
>
> I started with the 5 squares in the middle then added 4 arms on with $n - 1$ squares on each. $5 + 4(n - 1)$.



> How can you tell that it is diagram 31 that has 125 squares?
>
> If $4n + 1 = 125$, then $4n = 124$ and $n = 31$.
>
> The first diagram uses 5 squares, which leaves 120 squares. $120 \div 4 = 30$, so it must be the 31st diagram.

(ii) Generating more questions

There are many other questions an examiner might ask. Ask learners to suggest some of these and write them at the bottom of Sheet 1. In doing this, they should not change the diagrams in any way, but simply ask new questions about the existing diagrams.

> Can you have a cross diagram with 500 squares? How do you know?
>
> The first cross is 3 squares long. How long is the nth cross?
>
> The first diagram has a perimeter of 12. What is the perimeter of the 4th diagram? The 100th diagram? The nth diagram?
>
> Is it possible to draw a cross diagram with a perimeter of 100? How can you be sure?
>
> The first diagram can be made from 16 matchsticks. How many matchsticks would it take to make the 6th diagram? The 20th diagram? The nth diagram?
>
> Is it possible to make a cross diagram with 100 matchsticks? How can you be sure?

Working in groups (1)

Ask learners to choose one of these questions that they think they can answer and encourage them to work on it in pairs. Learners may like to compare their different ideas by writing them on overhead transparencies or on the board.

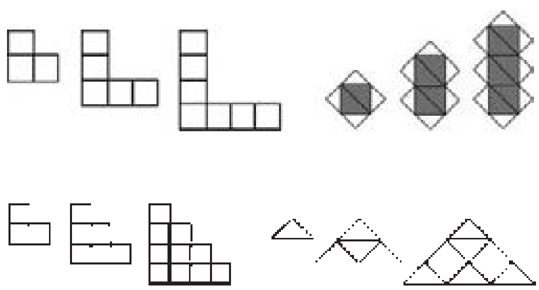
Whole group discussion (2)

(iii) Developing the situation

Hand out copies of Sheet 2 – *Template for growth patterns*. Explain to learners that they will work in pairs or threes to write their own GCSE question using this template.

Discuss with them how they might do this. They will need to draw a clear simple starting diagram, use a consistent rule for growth, and think of a suitable way of measuring the growth (squares, perimeters, areas, matchsticks etc.).

Many possibilities may be suggested, e.g. using different shapes and colours.



Linear and quadratic sequences may be developed.

Working in groups (2)

Learners should then work in pairs or threes and write new questions together with solutions (on the back of the sheet). Encourage learners to ask questions that they consider challenging but that are within their capabilities.

If you wish, you may hand out squared or triangular dotted paper for learners to use when trying to devise more imaginative questions.

The new questions should be passed around the groups to be answered by other learners. Where learners have difficulties in answering questions, the question writers should explain what they intended and act as a teacher, helping each other to answer the questions.

Alternatively, some of the new questions may be photocopied for future sessions or for homework.

Reviewing and extending learning

Finally, hold a whole group discussion on what has been learned, drawing out any common misconceptions. You should include a discussion of the level of difficulty of the new questions.

What learners might do next

If learners wish to explore generalisations of patterns further, using a computer, there is a useful program at www.fi.uu.nl

Ask learners to choose another question from an exam paper and follow the process adopted in this session, i.e.

(i) Answer the question.

(ii) Ask new questions about the same situation, and answer them.

(iii) Change the situation and make a new question.

Further ideas

This method for developing exam questions may be used in any topic. Examples in this pack include:

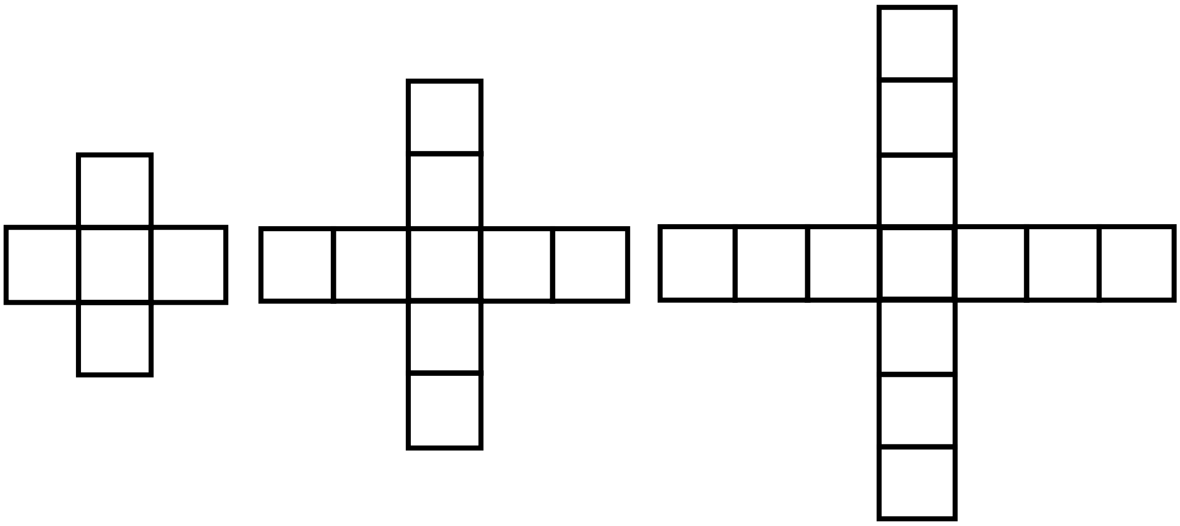
N10 Developing an exam question: number;

SS8 Developing an exam question: transformations;

S7 Developing an exam question: probability.

A8 Sheet 1 – *Growing cross patterns*

Some cross patterns are made of squares.



(a) How many squares will be in diagram 6?

(b) Write down an expression for the number of squares in diagram n.

(c) Which diagram will have 125 squares?

Write down some other questions that may be asked about this situation.

A8 Sheet 2 – *Template for growth patterns*

A pattern is made of. . . . .

(a) How many . . . . will be in diagram number . . . . ?

(b) Write down an expression for the number of . . . . in diagram n.

(c) Which diagram will have . . . .

(d) . . . .

. . . .

. . . .

. . . .

A9 • Performing number magic

Mathematical goals

To enable learners to:

- develop an understanding of linear expressions and equations;
- make simple conjectures and generalisations;
- add expressions, 'collecting like terms';
- use the distributive law of multiplication over addition in simple situations;
- develop an awareness that algebra may be used to prove generalisations in number situations.

Starting points

It is helpful, but not essential, if learners have previously encountered the idea of using letters to represent variables and already have some experience of simplifying expressions by 'collecting like terms'. These ideas are developed during the session.

The activity begins by inviting learners to look at some simple number magic. This is best done using the computer software *Number magic* that is provided on the DVD, but it can also be done using paper copies of the tricks.

The activity involves the following steps:

- learners familiarise themselves with a trick;
- learners try to work out how the trick is done, using algebra;
- learners try to improve the trick, making it more impressive.

Thus, this session meets the needs of all learners by allowing them to create tricks at different levels of difficulty.

Materials required

For each small group of learners you will need:

- calculator;
- computer running the software *Number magic*;

or at least one of:

- Sheets 1 to 5 – *Trick 1 to 5*

(with OHTs of these).

Time needed

From 1 to 2 hours, depending on the number of tricks used with each group of learners.

Suggested approach

Beginning the session

Begin the session by performing a 'lightning calculation' trick for the whole group. For example:

14
8
22
30
52
82
134
216
350
566
Total: 1474

Ask someone to suggest two whole numbers. Write these numbers one above the other at the top of the board. In the example, the two numbers are 14 and 8.

Now ask someone to write the sum of these two numbers underneath. They should then repeatedly add the last two numbers in the list and write the answer underneath. Continue in this way until 10 numbers are listed. Now challenge the group to add all ten numbers together.

As soon as the seventh number has been listed (in this case it is 134), you mentally multiply this number by 11 and write the answer (in this case 1474) on a large sheet of paper and, very obviously, stick it to the board so that the answer is hidden.

When learners arrive at an answer, turn over your sheet and display the answer. How did you know this before the list of numbers was complete?

Now explain how the trick is done, using algebra.

1	x
2	y
3	$x + y$
4	$x + 2y$
5	$2x + 3y$
6	$3x + 5y$
7	$5x + 8y$
8	$8x + 13y$
9	$13x + 21y$
10	$21x + 34y$
Total	$55x + 88y$

Ask learners to write x and y instead of the two starting numbers and get them to produce the rest of the table.

Emphasise why we can 'collect like terms'.

Finally, ask the group if they can spot any connection between the seventh term and the final total. Hopefully, they will see that $11(5x + 8y) = 55x + 88y$. This proves that the total will always be 11 times the 7th term.

Spend some time on the idea that algebra reveals the structure of a trick and proves that results work for all possible numbers.

This task involves two variables, x and y. The activities in this session start by requiring the use of only one variable, but they may be extended into two.

Introduce the main activity of the session using OHTs of the tricks on Sheets 1–5, or using an interactive whiteboard or data projector if you decide to use the computer programs.

Learners may enjoy working in pairs at a computer using the software provided. Alternatively, you could simply present the tricks on separate cards and ask learners to choose one or two to explore. Allow them to use calculators if they wish, as arithmetic is not our central concern here.

Explain to learners that, for each trick, they should:

- explore the trick, trying different numbers;
- work out how the trick is done. This usually involves spotting a connection between a starting number(s) and a finishing number. Algebra will be helpful here: "let the starting number be n and try to find an expression for the finishing number";
- improve the trick in some way.

Working in groups

Ask learners to work in pairs and give each pair one of the tricks.

Trick 1: Consecutive sum

In *Consecutive sum* learners vary the starting numbers and make conjectures about the totals produced. They may decide, for example that the final total is always a multiple of 5. Some may reason as follows:

> You add on a number one more, then you add a number two more, then a three more, then four more. That makes ten more altogether. So you add ten to five times the number.

Encourage learners to show this more formally, by writing n for the first number, $n + 1$ for the second and so on. The final total obtained is $5n + 10$ (or $5(n + 2)$). A quick way to predict the total from any starting number is to multiply by 5 and then add 10, or add 2 and then multiply by 5. Learners may be encouraged to develop this situation into a more complex number trick. It could be made more impressive by having more addends, for example.

Trick 2: Pyramid

In *Pyramid*, if the bottom left hand number is called x, then

- the bottom row is x, $x + 1$, $x + 2$, $x + 3$, $x + 4$;
- the second row is $3x + 3$, $3x + 6$, $3x + 9$;
- the top row is $9x + 18 = 9(x + 2)$.

So the short cut is simply to add 2 to the bottom left hand number and then multiply by 9 (or multiply by 9 and add 18). Learners may like to try creating larger Pyramids.

Trick 3: Routes

This simple trick uses the distributive law. The software allows learners to change the number in box A. If the number in box A is called x, then the number in box B is called $3x + 5$ and the number in box C is $3(x + 5) = 3x + 15$. This immediately shows that the number in box C is always 10 more than the number in box B. Learners should be encouraged to test this using large numbers and decimals in box A. Learners may like to explore the effect of changing the numbers along the routes.

Trick 4: Adding pairs

By experiment, learners may conjecture that, when you increase the first number by 1, the final number is increased by 3. They may draw up a table of results and look for patterns expressing these in algebra. Encourage them to prove their conjectures using algebra.

If the first number is called n, then successive numbers may be written: n, 10, $n + 10$, $n + 20$, $2n + 30$, $3n + 50$. The final number may therefore be obtained by multiplying the first number by 3 and adding 50. Learners may try improving the trick by generating longer sequences, or by changing the second number (10) to something different.

Trick 5: Calendar

In the *Calendar* software, learners drag the window to cover different sets of numbers.

n	$n + 1$	$n + 2$
$n + 7$	$n + 8$	$n + 9$
$n + 14$	$n + 15$	$n + 16$

$n - 8$	$n - 7$	$n - 6$
$n - 1$	n	$n + 1$
$n + 6$	$n + 7$	$n + 8$

In order to explain the trick, encourage learners to choose one of the numbers to be represented by, say, n and then represent the remaining numbers in the window in terms of n.

Thus, if the top left hand corner of the window is n, then the sum of eight numbers in the window border is $8n + 64$. The number in the centre of the window is $n + 8$.

Since $8(n + 8) = 8n + 64$, the sum of numbers in the window is 8 × the centre number. The trick is done by mentally dividing the total by 8.

Learners may like to try substituting n for different positions in the window. For example, if the centre number is called n, the surrounding numbers simply add to $8n$. Learners may like to try using different shaped windows on other grids.

Reviewing and extending learning

Ask learners to perform each of the magic tricks for the others in the group. They should then explain their analysis and show any improvements they have made.

What learners might do next

You could ask learners to use algebra to analyse a statement such as the following:

The sum of n consecutive numbers is divisible by n.
Is this always, sometimes or never true?
Can you say when it is true and when it is not?

A9 Sheet 1 – *Trick 1*

Consecutive sum

In this sum you add consecutive numbers.

The audience changes the top number and tells you what it is.

You immediately give the total.

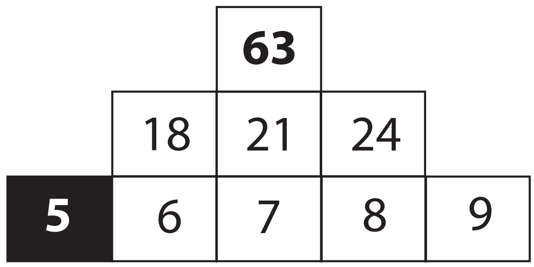
How is the trick done?

Can you make the trick more impressive?

	23
	24
	25
	26
+	27
Total	**125**

A9 Sheet 2 – *Trick 2*

Pyramid



The bottom row of this pyramid contains consecutive numbers.

Each other number is found by adding the three numbers beneath it.

The audience changes the bottom left hand number and tells you what it is.

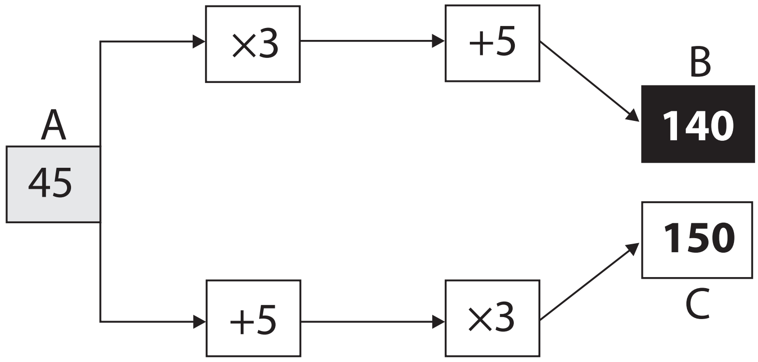
You immediately say what the top number is.

How is the trick done?

Try to make the trick more impressive.

A9 Sheet 3 – *Trick 3*

Routes



You are blindfolded.

The audience changes the number in box A.

The audience tells you the number in box B.

You immediately say the number in box C.

How is the trick done?

Can you make the trick more impressive?

(You can change the ×3 and +5 to other numbers if you want.)

A9 Sheet 4 – *Trick 4*

Adding pairs

5	**10**	15	25	40	**65**

Each number is the sum of the two previous numbers.

The audience changes the numbers in the two left hand boxes.

You immediately say the number in the right hand box.

How is the trick done?

Try to make the trick more impressive.

A9 Sheet 5 – *Trick 5*

Calendar

Mon	Tues	Wed	Thur	Fri	Sat	Sun
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

The audience moves the window to cover 8 dates.

They add the 8 dates together and tell you the total.

You immediately tell them the number in the middle of the window.

How is the trick done?

Try to make the trick more impressive.

Mon	Tues	Wed	Thur	Fri	Sat	Sun
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

A10 • Connecting perpendicular lines

Mathematical goals

To help learners to:

- identify perpendicular gradients;
- identify, from their equations, lines that are perpendicular;
- relate their learning about perpendicular lines to their previous learning about straight lines;
- explain the reasons why lines are parallel and perpendicular.

Starting points

Learners should:

- have some knowledge of equations of straight lines;
- be able to identify parallel lines;
- be able to use gradient triangles.

Materials required

For each learner you will need:

- mini-whiteboard;
- (possibly) a copy of Sheet 2 – *Perpendicular bisectors*.

For each small group of learners you will need:

- several sheets of squared paper;
- a protractor;
- Sheet 1 – *Properties, enlarged onto A3 paper*;
- Card set A – *Equations*;

optional

- graphic calculators.

Time needed

At least 1 hour 30 minutes.

Suggested approach

Beginning the session

Use mini-whiteboards to check that learners can remember how to calculate the gradient of a line between two given points.

Working in groups (1)

Ask learners to work in pairs. Give each pair some squared paper. Ask them to draw a line that has gradient 2. Next, ask them to draw a line that is perpendicular to this (using a protractor if necessary) and find its gradient. Suggest that they try other starting gradients and, working together, find the connection between a gradient and its perpendicular gradient.

Whole group discussion

Discuss the findings and check learners' understanding of gradients and perpendicular gradients by using mini-whiteboards and open questions such as:

> Give me an example of a line that has gradient 4.
>
> Give me an example of a line that is perpendicular to $y = 3x - 2$.
>
> Show me the equations of two lines that are perpendicular.

Working in groups (2)

Give each pair a copy of Sheet 1 – *Properties* (size A3) and Card set A – *Equations*. Ask them to match two equations to each property and add a property for the two that are left over.

The missing property is just passing through the point of intersection of the two lines. It is included so that learners cannot match the last two properties by default (i.e. there are only two cards left so they must fit into the last property).

The equations have been chosen to highlight the possible misconception that the number in front of x is the gradient and the y intercept is the number at the end. Therefore, while learners are working on matching, it is useful to ask them to explain their reasoning and to have graphic calculators or a computer with a graphical drawing package available to help sort out any problems.

If a pair finish early, give them some blank cards to write another equation for each property and ask them to write a justification as to why they match.

Reviewing and extending learning

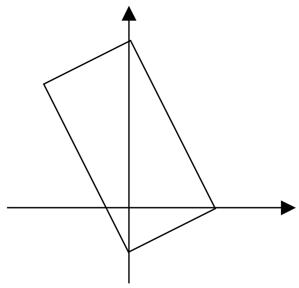
Discuss the matchings by asking questions such as:

Why were these two not parallel?

How do you know that these are perpendicular?

These two equations have both got a 4 at the end. Why do they not have the same y intercept?

Ask learners to find possible equations to make the shape:



Ask learners to generalise their findings using the equation $ax + by + c = 0$.

What learners might do next

Use Sheet 2 – *Perpendicular bisectors* to follow through a skeleton solution. Learners should write an explanation beside each step of the solution.

Further ideas

The activity of matching cards to properties can be used for any type of function.

It can also be used for properties of number, shape and space, and data handling.

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A10 Card set A – *Equations*

$y = 4x + 4$	$4y = x + 3$
$y = 8x - 3$	$y + 4x + 6 = 0$
$3y = 2x - 8$	$y + 6x = 11$
$y + 8x = 6$	$2y + 8 = 3x$
$2y + x = 4$	$2y = 8x + 3$
$y = 6x - 4$	$y + x + 8 = 0$

A10 Sheet 1 – *Properties*

These lines are parallel.	**These lines are perpendicular.**
These lines have the same y intercept.	**These lines have the same x intercept.**
These lines both go through the point (1, 5).	**These lines .....**

A10 Sheet 2 – *Perpendicular bisectors*

Question: Find the perpendicular bisector of the line joining the points (–2, 11) and (4, –7).

Follow the solution and explain what is happening at each stage.

Solution	Explanation
$-\frac{18}{6} = -3$	
$-3 \rightarrow \frac{1}{3}$	
$y = \frac{1}{3}x + c$	
$(1, 2)$	
$2 = \frac{1}{3} \times 1 + c$	
$c = \frac{5}{3}$	
$y = \frac{1}{3}x + \frac{5}{3}$	
$3y = x + 5$	

A11 • Factorising cubics

Mathematical goals

To enable learners to:

- associate x-intercepts with finding values of x such that $f(x) = 0$;
- sketch graphs of cubic functions;
- find linear factors of cubic functions;
- develop efficient strategies when factorising cubic functions.

Starting points

Learners should have some familiarity with quadratic graphs and factorising quadratic functions.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Factors*;
- Card set B – *True/false*.

Time needed

At least 1 hour 15 minutes.

Suggested approach

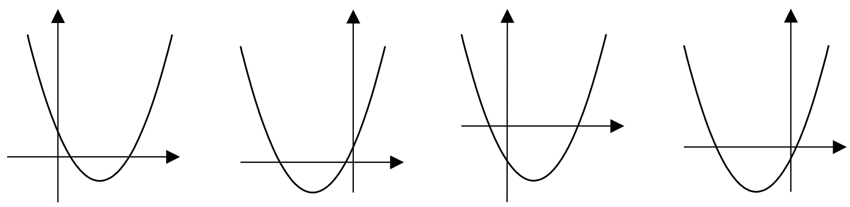
Beginning the session

Give out mini-whiteboards and ask learners to sketch a few straight lines such as $y = 4x - 8$ and $2y = 5x - 10$, marking the x and y intercepts. Through discussion, ensure that learners find the y intercept by putting x equal to zero and the x intercept(s) by putting y equal to zero.

Revise quadratic graphs by asking learners to sketch the graph of $y = (x - 3)(x + 4)$. Discuss how the x intercepts and y intercept were found.

Practise a few more examples.

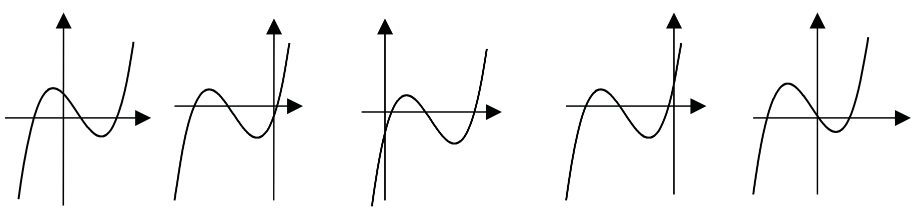
Then draw quadratic graphs on the board and ask for possible equations. Graphs could be:



Pay particular attention to the position of the intercepts.

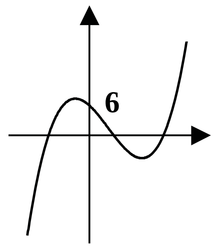
Then ask learners to sketch $y = (x - 1)(x - 2)(x - 3)$ on their whiteboards, using the same ideas (as they used with the quadratic functions) of finding the intercepts first. Check that everyone understands by asking learners to explain their graphs. Repeat for another two or three examples using a variety of + and – in the brackets.

Ask learners to sketch functions such as $y = (x - 2)(x^2 + 7x + 12)$. Discuss the usefulness of factorising. Then ask learners to suggest equations for graphs such as:



Use some of the suggestions to consider intercepts again. Then ask learners to suggest possible equations for graphs with a y intercept marked.

For example:



Repeat for a few more examples, discussing the need for the choice of brackets to give the correct y value when $x = 0$.

Ask learners what they can say about the sketch of $y = x^3 - 7x^2 + 4x + 12$ (e.g. y intercept is 12) and lead the discussion on to the need to be able to factorise cubic functions in order to sketch graphs. Ask for suggestions for factors and how they can be tested (values of x that make the brackets zero should also make y zero and making x equal to zero should give the same value of y in both the original equation and the factorised form).

Working in groups

Ask learners to work in pairs. Give Card set A – *Factors* to each pair and ask them to lay the cards out on the table. For a larger group of learners, provide two copies of Card set A, combined into a single set.

Write a cubic function on the board such as $y = x^3 - 2x^2 - 19x + 20$ and ask each pair/group to find as many sets of factors as they can (without checking any, other than giving the correct y intercept). Write a few sets on the board and ask for comments about whether they are appropriate ones to test.

Test one factor. Then ask the pairs if they want to change any of their set of three factors in the light of the result. Test another factor. Allow pairs to change factors again if they want to. Repeat until all the pairs have the correct set of three factors. Repeat for another cubic function such as $y = x^3 - 9x^2 + 26x - 24$.

Give each pair a cubic function that will factorise. Ask them to test values of x until they can come up with three factors. Suggest that it might be more convenient to use function notation to record their work. (This suggestion could have been made earlier.) When they have found all three factors, they should write them on the board. Give them another cubic function to work on, with the instruction that they should look at the completed ones to find strategies that make the process more efficient.

Learners who find the task difficult could be given functions that have positive roots.

Learners who find the task easy could be given a more challenging function for their second one, e.g. one with repeated roots, or passing through the origin.

Whole group discussion

Discuss the strategies that have been used and consider which are most efficient.

Reviewing and extending learning

Give each pair/group Card set B – *True/false*. They have to sort each pair of cards into 'True' and 'False' and be prepared to justify their answers. In whole group discussion, check that all learners understand the principles.

What learners might do next

This session could be followed by work on the remainder theorem and algebraic division.

A11 Card set A – *Factors* (page 1)

$(x - 1)$	$(x - 2)$	$(x - 3)$
$(x + 1)$	$(x + 2)$	$(x + 3)$
$(x - 4)$	$(x - 5)$	$(x - 6)$
$(x + 4)$	$(x + 5)$	$(x + 6)$
$(x - 7)$	$(x - 8)$	$(x - 9)$
$(x + 7)$	$(x + 8)$	$(x + 9)$
$(x - 10)$	$(x + 1)$	$(x + 2)$
$(x + 20)$	$(x + 1)$	$(x + 2)$
$(x - 12)$	$(x + 1)$	$(x - 2)$
$(x - 20)$	$(x + 1)$	$(x - 2)$

A11 Card set A – *Factors* (page 2)

$(x - 24)$	$(x - 1)$	$(x + 3)$
$(x + 24)$	$(x - 1)$	$(x + 6)$
$(x + 10)$	$(x - 1)$	$(x + 5)$
$(x + 12)$	$(x - 1)$	$(x + 4)$
$(x - 1)$	$(x - 2)$	$(x - 3)$
$(x + 1)$	$(x + 2)$	$(x + 3)$
$(x - 4)$	$(x - 5)$	$(x - 6)$
$(x + 4)$	$(x + 5)$	$(x + 6)$
$(x - 1)$	$(x - 2)$	$(x - 3)$
$(x + 1)$	$(x + 2)$	$(x + 3)$
$(x - 4)$	$(x - 5)$	$(x - 6)$

A11 Card set B – *True/false* (page 1)

A1

$f(2) = 0 \Rightarrow$
$(x + 2)$ **is a factor of** $f(x)$

A2

$f(2) = 0 \Rightarrow$
$(x - 2)$ **is a factor of** $f(x)$

B1

$f(-3) = 0 \Rightarrow$
$(x + 3)$ **is a factor of** $f(x)$

B2

$f(-3) = 0 \Rightarrow$
$(x - 3)$ **is a factor of** $f(x)$

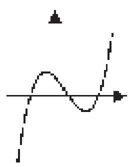
C1

$f(4) = 7 \Rightarrow$
$(x - 4)$ **is a factor of** $f(x)$

C2

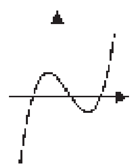
$f(4) = 7 \Rightarrow$
$(x - 4)$ **is not a factor of** $f(x)$

D1



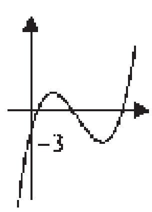
Possible equation for this graph is:
$f(x) = (x - 2)(x - 5)(x - 1)$

D2



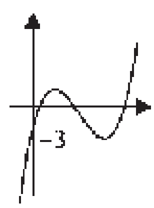
Possible equation for this graph is:
$f(x) = (x - 2)(x + 5)(x - 1)$

E1



Possible equation for this graph is:
$f(x) = x^3 + 6x^2 + 7x - 3$

E2



Possible equation for this graph is:
$f(x) = x^3 + 6x^2 + 7x + 3$

A11 Card set B – *True/false* (page 2)

F1 $(x - 3)$ is a factor of $f(x) = x^3 - x^2 - 3x - 2$	**F2** $(x - 3)$ is not a factor of $f(x) = x^3 + x^2 - 3x + 2$
G1 If $f(x)$ is a cubic function and if $f(1) = 0, f(3) = 0$ and $f(0) = 12$ then $f(4) = 0$	**G2** If $f(x)$ is a cubic function and if $f(1) = 0, f(3) = 0$ and $f(0) = 12$ then $f(-4) = 0$
H1 If $f(x) = x^3 - 6x^2 - x + 6$ and $f(6) = 0$ then it would be a good idea to test $f(3)$	**H2** If $f(x) = x^3 - 6x^2 - x + 6$ and $f(6) = 0$ then it would be a silly idea to test $f(3)$

A12 • Exploring trigonometrical graphs

Mathematical goals

To give learners practice in:

- recognising translations, stretches and reflections of trigonometrical graphs from their equations;
- sketching trigonometrical graphs.

To introduce learners to:

- the period and amplitude of a trigonometrical graph.

Starting points

Learners should be familiar with the graphs of $y = \sin x$ and $y = \cos x$.

Learners should have previously met the basic transformations of functions, i.e. one-way stretches, translations and reflections, but not necessarily in the context of trigonometrical graphs.

Materials required

For each learner you will need:

- mini-whiteboard;
- Sheet 2 – *Trigonometrical graphs*.

For each small group of learners you will need:

- Card set A – *Trigonometrical equations*;
- some blank paper or card;
- Sheet 1 – *Properties of trigonometrical graphs, enlarged to A3 size*;

and optionally either:

- a computer with graph-drawing software;

or

- graphic calculators.

Time needed

At least 45 minutes.

Suggested approach

Beginning the session

Remind learners what the graphs of $y = \sin x$ and $y = \cos x$ look like and discuss the important features.

Working in groups (1)

Ask learners to work in pairs; give out Card set A – *Trigonometrical equations* to each pair of learners. Ask them to sort the cards into two sets, deciding for themselves the criteria for their sets. Then ask them to sort their largest set into two subsets, again choosing their own criteria. Ask them to repeat this once more for their largest set. They will now have four sets of cards. Their criteria might be as simple as 'Has a 2 in front' but that is fine.

Give out some blank paper or card. Ask learners to write a description of each of their sets and add an equation of their own to each set.

Whole group discussion (1)

Share all the criteria that learners have come up with and translate them into mathematical language if necessary; for example, "they have a 3 in front" can become "a one-way stretch in the direction of the y-axis with scale factor 3"; "A number in the bracket with x" can become "a translation of . . . in the direction of the x-axis".

If learners are not familiar with the link between transformations of a graph and its equation, use a computer with graph-drawing software or graphical calculators to justify the descriptions.

If learners need more practice with transformations and trigonometrical graphs, use mini-whiteboards to ask questions such as:

> Give me the equation of $y = \sin x$ after it has been reflected in the x-axis.
>
> Give me the equation of $y = \cos x$ after it has been stretched with a scale factor of 2 in the direction of the x-axis.
>
> $y = \sin x$ is transformed into $y = 4 \sin x$. What was the transformation involved?

Define 'period' and 'amplitude' and link these with stretches. Check learners' understanding of how the equation gives information about period and amplitude by asking them to find cards that fit certain criteria such as:

> Find me the equations of all the graphs that have an amplitude 3.

Find me the equations of two graphs that have the same period.

Find me the equation of a graph that has period 180°.

etc.

Working in groups (2)

Give out Sheet 1 – *Properties of trigonometrical graphs (enlarged)* to each pair of learners and ask them to find cards from Card set A to fit as many boxes as possible. When they have found a possible card they should place it in the right box on Sheet 1, covering up the property. They should aim to cover up as many boxes as possible.

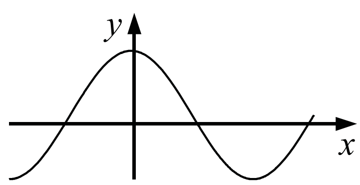
If learners finish early, ask them to find out if any of their remaining cards from set A fit into a box on Sheet 1. If there are some they cannot fit anywhere, ask them to extend the grid and write properties for those cards. If they have any boxes for which they cannot find a card they should write one for it.

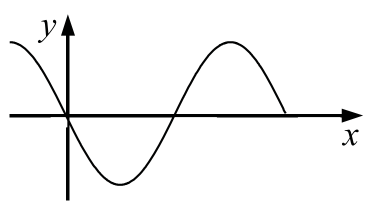
Whole group discussion (2)

It is likely that different pairs of learners will have placed different cards on a particular box. Choose one of the boxes and write down all the different cards that have been placed on it. Discuss as a whole group whether they are all appropriate and, if so, why. Repeat this for a few other boxes.

Give out mini-whiteboards and ask questions such as:

- Sketch me the graph of $y = 3\sin x$ or $y = \sin 2x$ or $y = \cos(x + 90)$
- Give me possible equations for





etc.

Reviewing and extending learning

Give out Sheet 2 – *Trigonometrical graphs* to each learner. Ask them to decide which equation goes with which graph. When they have done this, they should label the x and y axes accordingly.

Learners could then generalise their work by sketching and labelling graphs such as $y = a \sin x$ and $y = \cos bx$.

What learners might do next

Learners could consider how the axes of trigonometrical graphs would be labelled if the equations were using radians.

Further ideas

Sorting cards according to criteria chosen either by learners or by the teacher can be used for other functions such as quadratic or linear functions or circles. It can also be used for properties such as number types or shapes.

A12 Card set A – *Trigonometrical equations*

$y = 2\sin x$	$y = \sin(x + 90)$	$y = \cos x - 1$
$y = -\cos x$	$y = \sin x + 1$	$y = \cos(x + 90)$
$y = \sin(x - 90)$	$y = \cos(x + 180)$	$y = \cos 3x$
$y = \sin \frac{1}{2} x$	$y = \sin 2x$	$y = 3\cos x$
$y = 2\cos x$	$y = -\sin x$	$y = \cos(x - 180)$
$y = \frac{1}{2}\cos x$	$y = 3\sin x$	$y = \cos 2x$

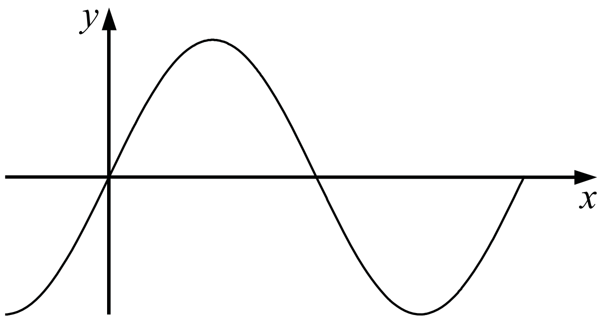
A12 Sheet 1 – *Properties of trigonometrical graphs*

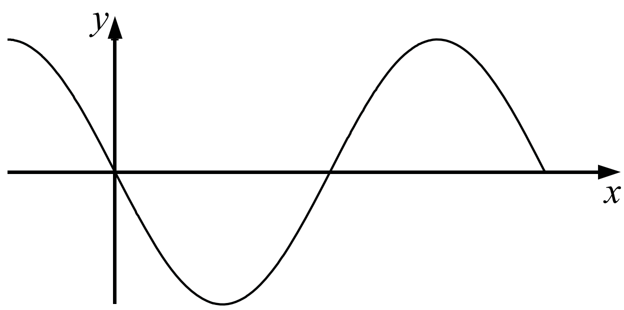
Period 180° **Passes through (0, 0)**	**Amplitude 3** **Passes through (0, 0)**
Period 360° **Passes through (0, 1)**	**Amplitude 1** **Passes through (0, –1)**
Period 180° **Passes through (0, 1)**	**Amplitude 1** **Passes through (90, 0)**
Period 360° **Passes through (0, –1)**	**Amplitude 3** **Passes through (–90, 0)**
Period 360° **Passes through (0, 0)**	**Amplitude 2** **Passes through (0, 0)**
Period 720° **Passes through (0, 0)**	**Amplitude 2** **Passes through (90, 0)**

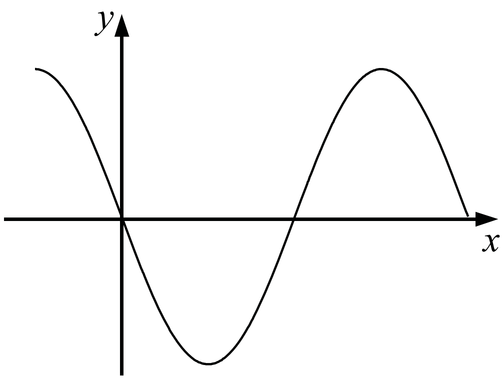
A12 Sheet 2 – *Trigonometrical graphs*

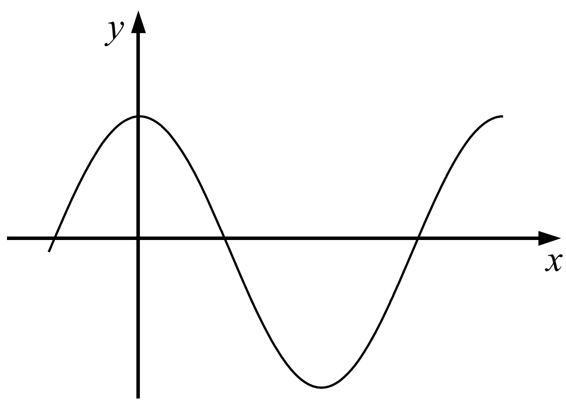
Match the equations to the graph and label the axes appropriately.

$y = -\sin x$ $\qquad$ $y = 3 \sin x$ $\qquad$ $y = \cos(x + 90)$ $\qquad$ $y = \cos 2x$









A13 • Simplifying logarithmic expressions

Mathematical goals

To enable learners to:

- develop their understanding of the laws of logarithms;
- practise using the laws of logarithms to simplify numerical expressions involving logarithms;
- apply their knowledge of the laws of logarithms to expressions involving variables.

Starting points

Learners should have some knowledge of the laws of logarithms as applied to numerical expressions.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Logarithms* (3 pages);
- Card set B – *Odd one out*.

Time needed

At least 45 minutes.

Suggested approach

Beginning the session

Start with some 'true or false?' questions to reinforce previous work on logarithms.

Possible statements could include:

$\log_2 8 = 4$ $\quad\quad$ $\log_2 32 = 5$

$\log_{16} 4 = \frac{1}{4}$ $\quad\quad$ $\log_3 7 + \log_3 5 = \log_3 12$

$\log_2 30 - \log_2 5 = \log_2 6$ $\quad\quad$ $\log_2 \frac{1}{8} = -3$

$3\log_7 4 = \log_7 12$ $\quad\quad$ $\log_5 6 + \log_4 7 = \log_5 42$

Learners could discuss each question in groups of two, three or four and write their agreed answer on a mini-whiteboard which they show on request. Ask them to justify their answers.

Working in groups (1)

Give each small group of learners Card set A – *Logarithms*. These cards are triangles which can be arranged into a hexagon so that the expressions on adjacent sides of the triangles are equal. Ask learners to arrange the cards into a hexagon.

As learners are piecing the hexagon together, jot down on the board any problems or interesting points that come out of their discussions.

If learners find this activity easy, you could replace two or three of the printed triangles with blanks so that learners have to write expressions on them to create matching sides. Alternatively, they could write expressions to match the sides of the hexagon.

For learners who find this activity difficult, you could mark or delete the outer sides of the hexagon and/or the vertices that go into the centre.

Whole group discussion

When learners have finished the activity, discuss the points you have written on the board. Also ask learners to explain why they matched certain sides, focusing particularly on the ones with algebraic expressions.

Reviewing and extending learning

Give learners Card set B – *Odd one out* cut into horizontal strips. You could give the strips out one at a time or all together. Ask learners, working in pairs, to identify which is the odd one out in each strip. When they have done this, they should write in the blank space as many expressions as they can think of that are equivalent to the odd one out. Discuss some of the possibilities by asking learners to suggest an equivalent expression and then explain why it is equivalent.

Using mini-whiteboards, ask learners to give possible numbers or variables for the blanks in the statements below.

$$\log_{\square} \square = 3$$

$$\log_{\square} \square + \log_{\square} \square = 24$$

$$\square \log_{\square} \square = \log_{\square} \square$$

$$\log_{\square} \square - \log_{\square} \square = 6$$

What learners might do next

Solve equations such as $ab^x = c$ using logarithms.

Further ideas

The hexagonal jigsaws in this session can be adapted for many topics such as brackets, simplifying algebraic expressions, differentiation and integration, negative numbers, and multiplication tables. The software for producing your own hexagonal jigsaws is included on the DVD-ROM/CD in this pack.

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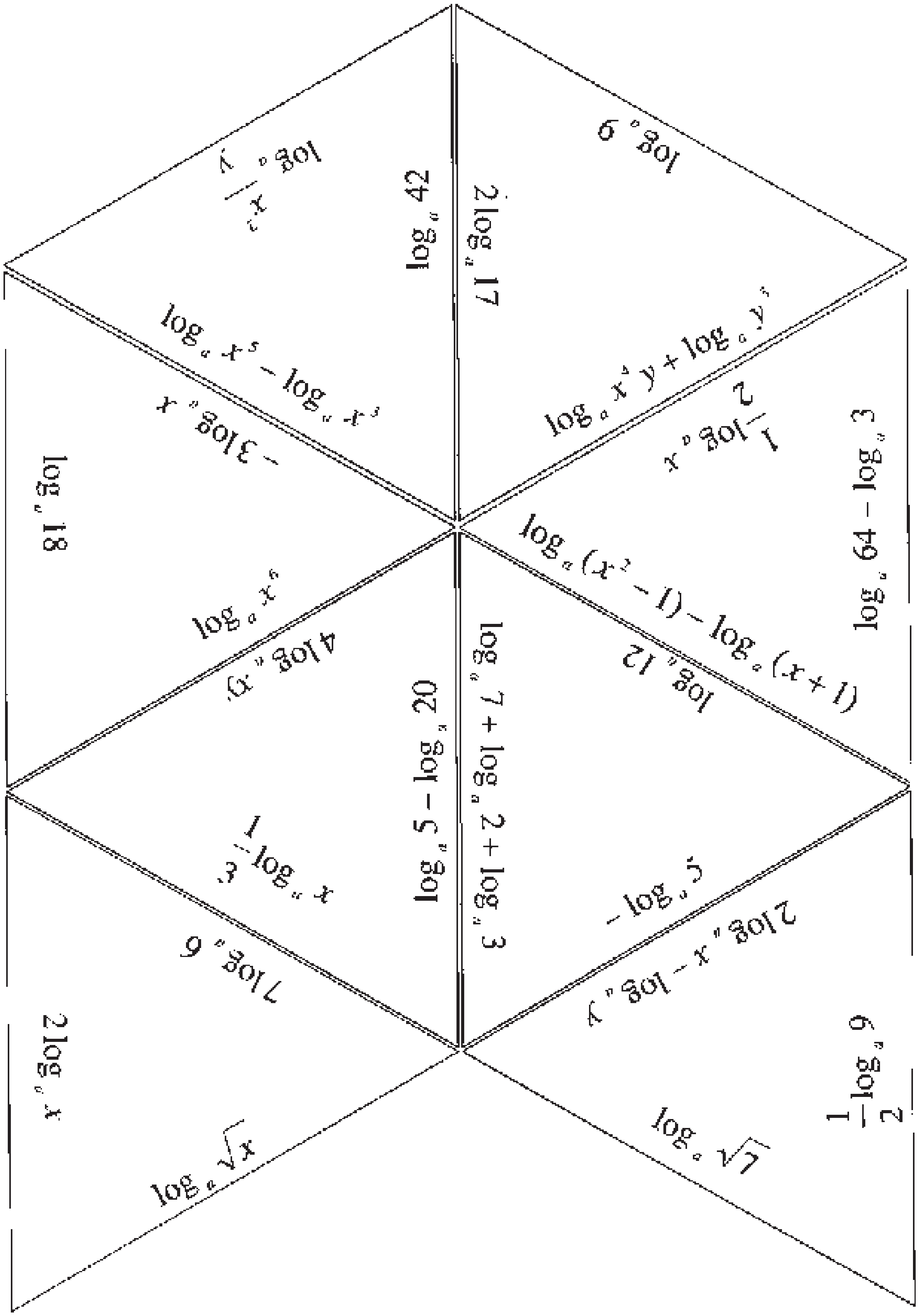
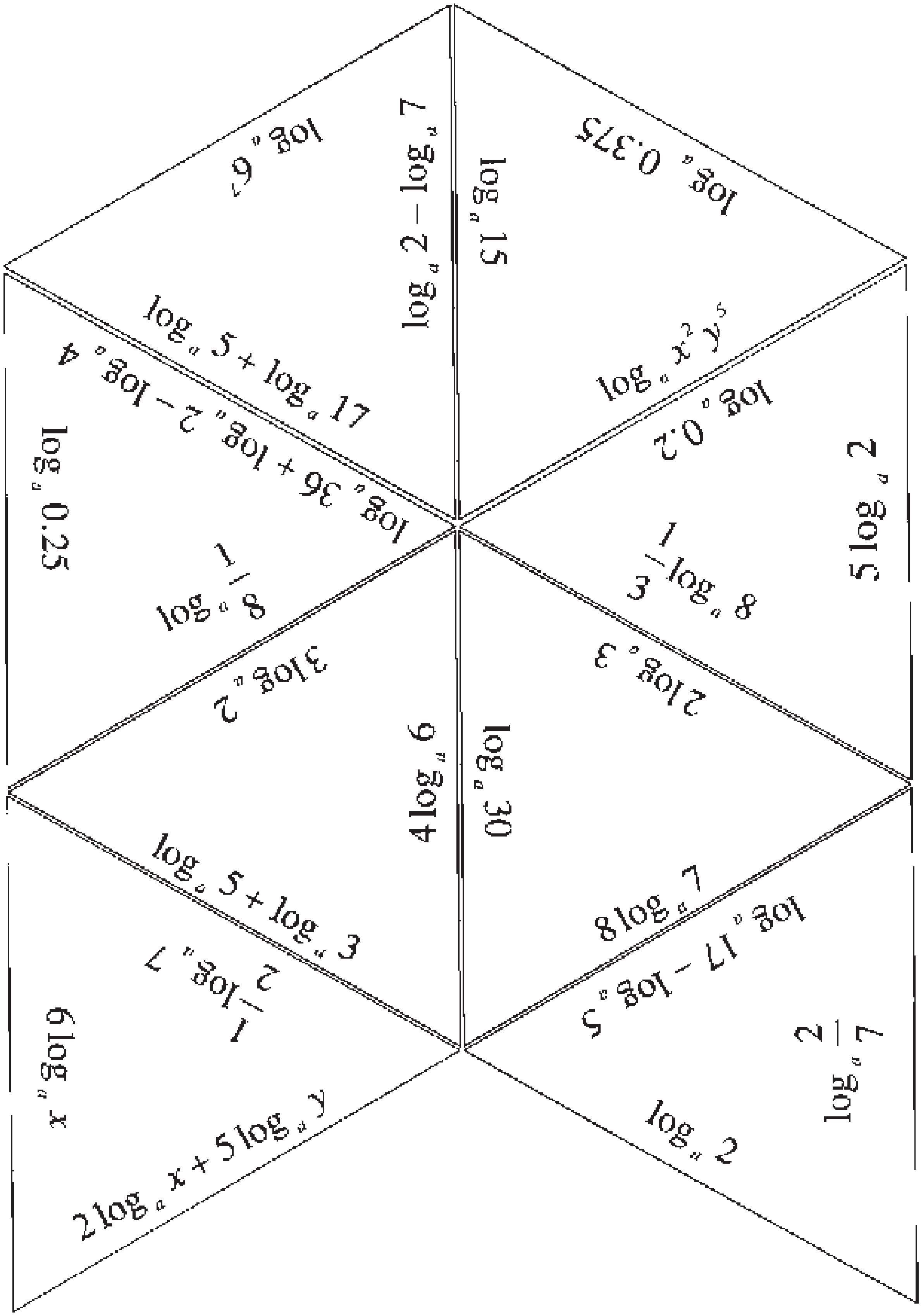
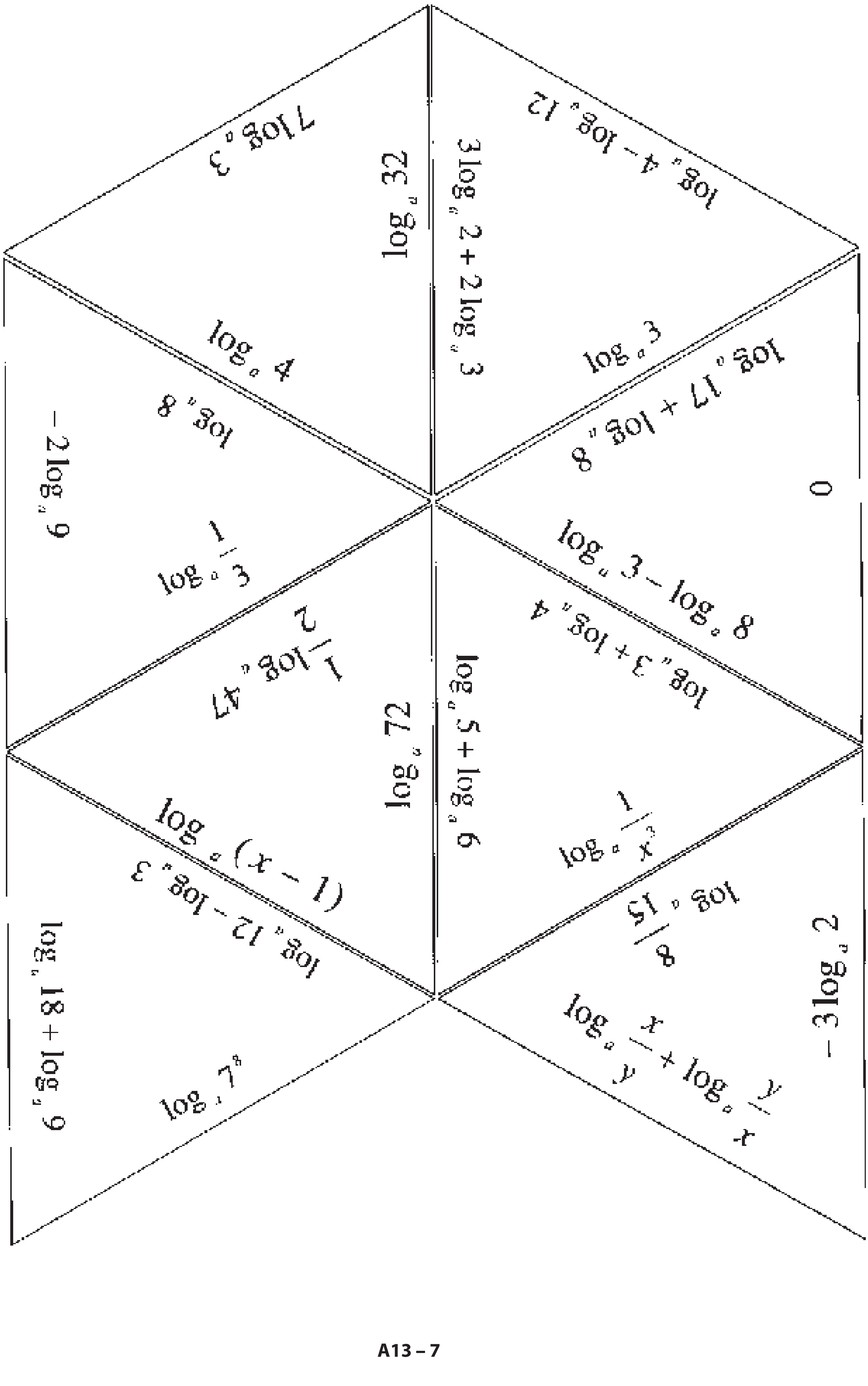
A13 Card set A – *Logarithms* (part 1)



A13 Card set A – *Logarithms* (part 2)



A13 Card set A – *Logarithms* (part 3)



A13 Card set B – *Odd one out*

$\log_2 8$	$\log_3 9$	$\log_4 64$	
$\log_2 0.5$	$\log_4 0.25$	$\log_8 0.5$	
$\log_2 8^3$	$\log_2 4^4$	$\log_2 2^9$	
$\log_2 x^6$	$\log_2 x^2 + \log_2 x^3$	$\log_2 x^2 + \log_2 x^4$	
$-2 \log_2 x$	$\log_2 \frac{1}{x^2}$	$\log_2 \sqrt{x}$	

A14 • Exploring equations in parametric form

Mathematical goals

To enable learners to:

- find stationary points when a function is given in parametric form;
- determine the nature of stationary points when the function is given in parametric form;
- find the intercepts when a function is given in parametric form.

Starting points

Learners should be able to find stationary points and their nature when equations are given in Cartesian form. Learners should understand the parametric form of an equation in the sense of being able to find pairs of coordinates for given values of the parameter. They should also know how to find $\frac{dy}{dx}$ from an equation given in parametric form.

Materials required

OHT 1 – *Graphs 1*

OHT 2 – *Graphs 2*

These could be shown on an interactive whiteboard.

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Graphs* (2 pages);
- Card set B – *Equations*;
- Sheet 1 – *Where are the errors?*;
- at least one large sheet of paper for making a poster;
- felt tip pen;
- glue stick.

Time needed

At least 1 hour 15 minutes.

Suggested approach

Beginning the session

Show OHT 1 – *Graphs 1* on the board, hand-drawn on the board or via an OHP or a data projector. Ask learners to describe what is the same and what is different about these two graphs.

Then put OHT 2 – *Graphs* 2 on the board. This time ask learners, working in groups of two or more, to discuss the similarities and differences between the two graphs (one graph is the thick lines and the other is the thin lines). Ask for suggestions and write them briefly on the board. Then go through the list asking learners what sort of mathematics would be used to investigate that difference, e.g. "One graph is steeper than the other for a certain range of values of x" becomes "Investigate the gradient via $\frac{dy}{dx}$". Encourage learners to comment on the shapes of the graphs and to suggest what happens when x is large, for example.

If necessary, check, using mini-whiteboards, that learners remember how to differentiate equations in parametric form.

Working in groups

Ask learners to work in pairs. Explain that they are going to be given five equations in parametric form and five graphs. Their task is to match each graph with its equation and to give as much justification as they can for their decision. Encourage them to use some of the criteria that they suggested earlier. Explain that they should use their knowledge of dealing with Cartesian equations and work out how to apply it to parametric equations. All working should be shown on the large sheets of paper. The pairs of graphs and equations should be stuck alongside the working.

Give Card set A – *Graphs* and Card set B – *Equations* to each pair of learners, plus a large sheet of paper, felt tip pen and glue stick.

The task could be set at different levels of challenge. 'Normal' challenge could be graphs B, C, D and E along with their equations; 'medium' challenge could be all five graphs and five equations; 'high' challenge could be graphs B, C, D and E with all the equations and learners have to sketch the missing graph.

As learners investigate their functions they may get confused between t, x and y. Some may try to identify the stationary point at a value of t instead of x. It is helpful to go round and ask learners to guide you through what they are doing and why they have come to the conclusion that they did. It is also a good opportunity for learners to see that:

- not all curves have stationary points and the algebra can show this;
- important properties about the shape of a curve can be found by looking at t approaching zero or infinity.

Whole group discussion

When all pairs have matched equations to graphs with some justification, share the findings. Ask each pair of learners to identify something about one of their graphs that surprised them and how the algebra showed them what was actually happening. Encourage those who have used non-standard reasons for justifying their matching to share these reasons.

Reviewing and extending learning

Give each pair of learners Sheet 1 – *Where are the errors?*. Ask them to mark the answers and identify any errors. When learners have finished, discuss the errors and clarify the misconceptions.

What learners might do next

Find equations of tangents and normals using equations in parametric form.

Further ideas

Matching equations and graphs and justifying decisions can be used for all types of functions. It allows learners to answer at different levels.

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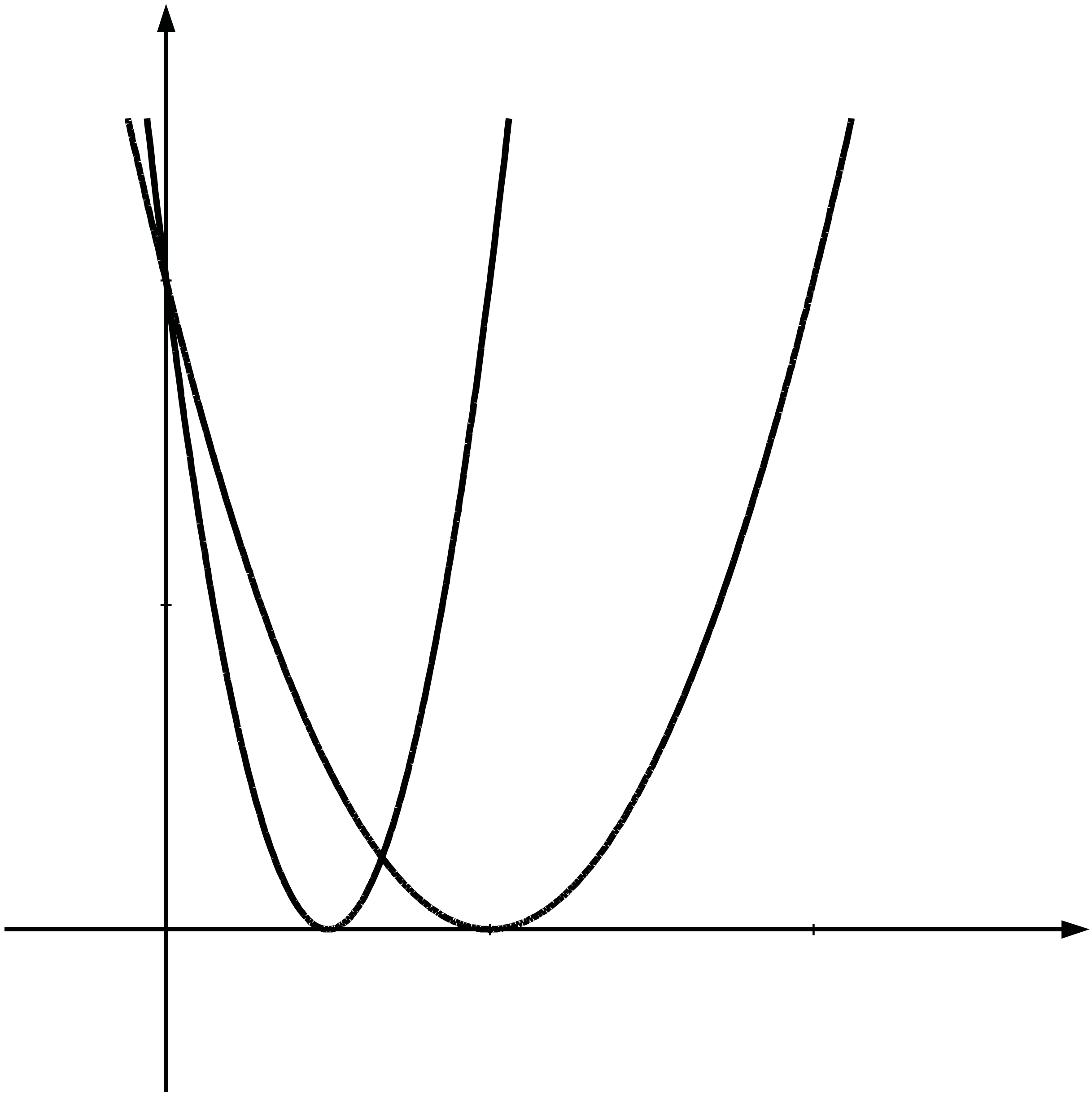
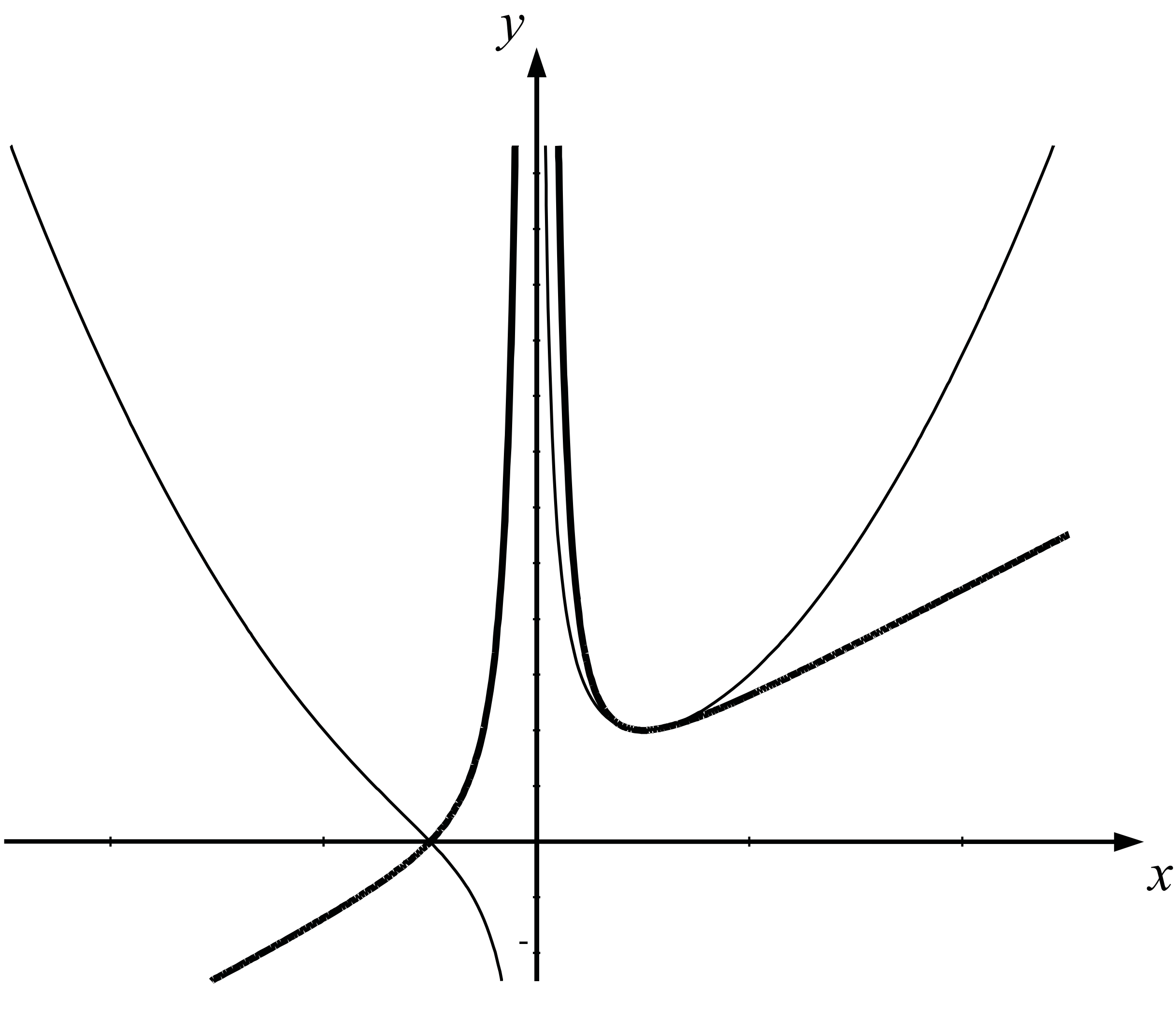
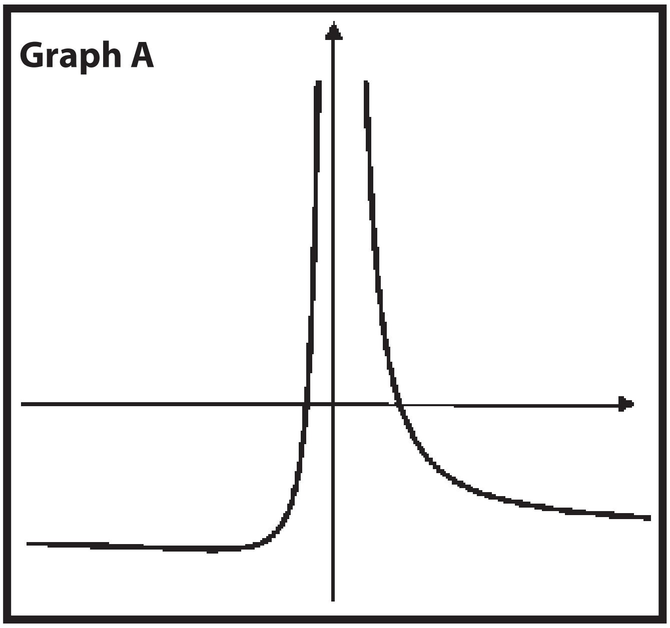
A14 OHT 1 – *Graphs 1*

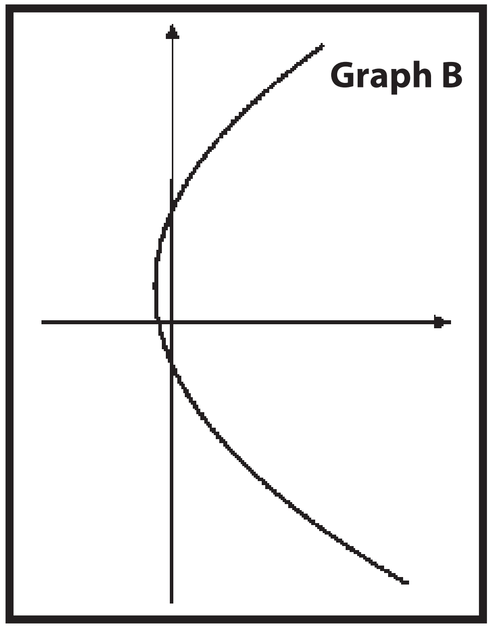


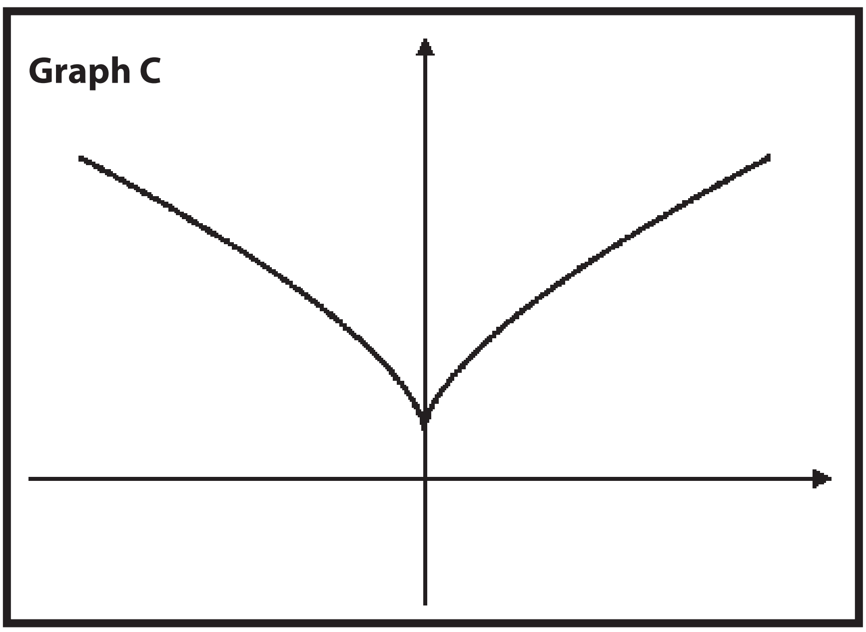
A14 OHT 2 – *Graphs 2*



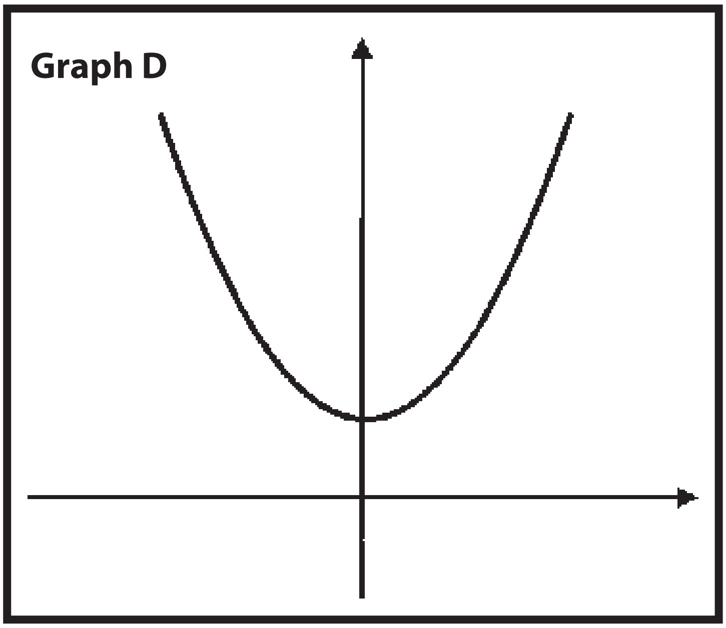
A14 Card set A – *Graphs and equations*

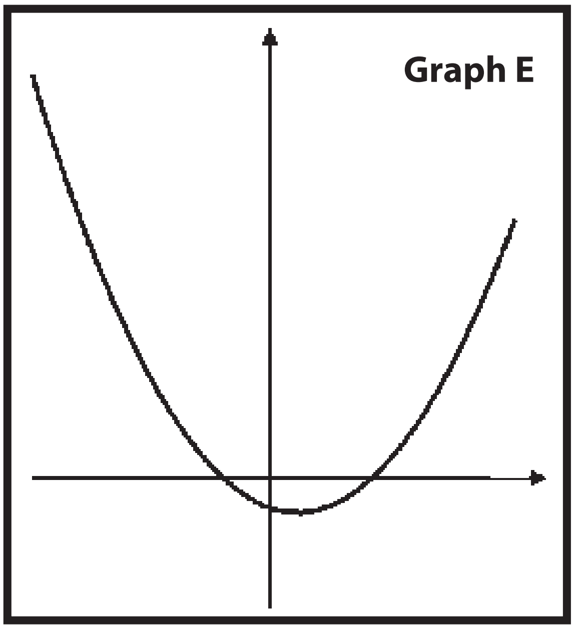






A14 Card set A – *Graphs and equations* (continued)





A14 Card set B – *Equations*

$x = 3t + 2$	$y = t^2 + t$
$x = t^3$	$y = t^2 + 1$
$x = \frac{1}{t}$	$y = t^2 + t - 2$
$x = 0.5t$	$y = t^2 + 1$
$x = t + t^2$	$y = 3t + 2$

A14 Sheet 1 – *Where are the errors?*

Questions

1. Find the intercepts of the graph whose equation is: $x = t^2 - 1, \quad y = t - 6$

2. Find the gradient of the curve: $x = t^3, \quad y = 2t^2 - 1$
at the point where $t = 3$.

3. Find the gradient of the curve: $x = \frac{12}{t^4}, \quad y = 3t + 1$
at the point (3,7).

4. Find any stationary points on the curve: $x = 2t^2 - 1, \quad y = t^3 + 1$

Answers

1.

$x = t^2 - 1$
$x = 0 \Rightarrow t^2 - 1 = 0$
$\Rightarrow t^2 = 1$
$\Rightarrow t = 1$
$\Rightarrow y = -5$

$y = t - 6$
$y = 0 \Rightarrow t - 6 = 0$
$\Rightarrow t = 6$
$\Rightarrow x = 35$

2.

$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 4t$

$\frac{dy}{dx} = \frac{3t^2}{4t} = \frac{3 \times 3^2}{4 \times 3} = \frac{27}{12} = 2.25$

3.

$y = 3t + 1 = 7 \quad \Rightarrow \quad t = 2$

$\frac{dy}{dt} = 3$

$\frac{dx}{dt} = 4 \times \frac{12}{t^3}$

At $t = 2, \frac{dx}{dt} = 4 \times \frac{12}{8} = 6$

$\frac{dy}{dx} = \frac{3}{6} = 0.5$

4.

$\frac{dy}{dt} = 3t^2 \qquad \frac{dx}{dt} = 4t$

$\frac{dy}{dx} = \frac{3t^2}{4t} = \frac{3t}{4} = 0 \quad \Rightarrow \quad t = 0$

$\therefore$ *Stationary point is at (0,1)*

SS1 • Classifying shapes

Mathematical goals

To help learners:

- name and classify polygons according to their properties;
- develop mathematical language to describe the similarities and differences between shapes;
- develop convincing explanations as to why combinations of particular properties are impossible.

These goals may be adapted for learners aiming at lower levels. For example, you may decide to focus on just the first two goals.

Starting points

No prior learning is needed.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Shapes*;
- some blank cards;
- Sheet 1 – *Classifying by symmetry*;
- Sheet 2 – *Classifying by regularity*;
- Sheet 3 – *Classifying triangles*;
- Sheet 4 – *Classifying quadrilaterals*;
- Sheet 5 – *Classifying by perimeter and area*.

The whole group discussion will be easier if you make OHTs of Card set A – *Shapes* and of the five sheets.

For learners aiming at lower levels, you may decide to begin by using just one-dimensional classification rather than the two-dimensional grids.

Time needed

Approximately 1 to 2 hours, depending on how many classification grids (sheets) are used.

Suggested approach

Learners aiming at lower levels may do only the first sort, with two descriptions.

Beginning the session

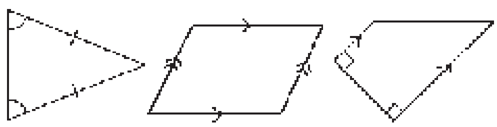
Ask learners to work in pairs. Give each pair of learners Card set A – *Shapes*. Ask them to sort the shapes into two groups using criteria of their own choice. Next, ask them to sort each group into two, using further criteria. Give out blank cards and ask learners to write a description of each of their four groups and also to draw another shape to add to each group.

Whole group discussion: reviewing names and notation

Ask learners to share their criteria for sorting the shapes. Show how their four groups may be displayed using two-way tables. Help them to translate what they say into 'official' mathematical language such as:

- names of polygons (triangle, rhombus, regular etc.);
- names of angles (interior, exterior, acute, obtuse, reflex);
- terms for symmetry (line, rotational);
- terms that relate to lines (adjacent, equal, parallel, perpendicular).

Describe the notations that are commonly used to describe pairs of equal lengths, equal angles, right angles and parallel sides. Ask learners to label some shapes in this way. For example:



Working in groups

Ask learners to work in pairs. Give each pair one of the Sheets 1 to 5. Ask them to place shapes into appropriate cells. Sometimes, several shapes may go in a cell. If learners feel that a cell is impossible to fill, they should explain why this is so.

Learners who struggle may be asked to find shapes corresponding to one criterion at a time (e.g. "Regular or irregular?"). When they have done this, they might then be encouraged to use two-way classifications such as those found on the grids.

Learners who find the task straightforward should be pressed for clear, written explanations as to why certain combinations of criteria are incompatible. This can be very challenging.

Listen to learners' explanations. Note obvious misconceptions that emerge for the final whole group discussion. For example, many learners assume that a parallelogram has a line of symmetry.

Reviewing and extending learning

Using mini-whiteboards, ask learners to show examples of:

- a quadrilateral with two lines of symmetry;
- a triangle with three lines of symmetry;
- a right angled isosceles triangle;
- a triangle with all acute angles;
- a shape whose interior angles add up to 360°;
- a trapezium with only one right angle (impossible!);
- a quadrilateral with one reflex angle;

. . . and so on.

What learners might do next

There are of course many other ways of classifying shapes. You may like to suggest that learners invent methods of their own. For example, they could try to draw a table showing 'number of lines of symmetry' against 'order of rotational symmetry'. This is quite hard to fill in, as there are many impossible entries.

Further ideas

Classification activities are very powerful and can be used across the curriculum. For example, you could ask learners to classify and name sets of numbers, graphs, equations and so on.

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SS1 Card set A – *Shapes* (page 1)

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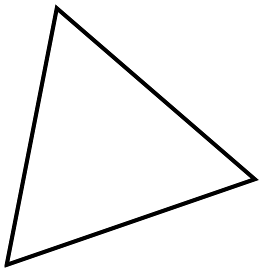

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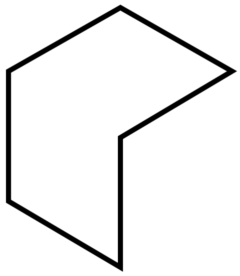

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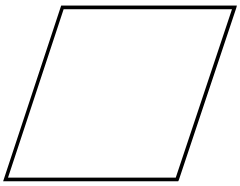

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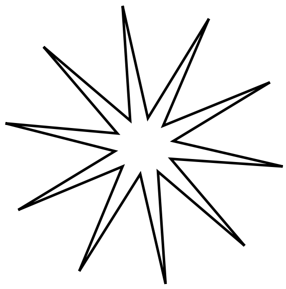

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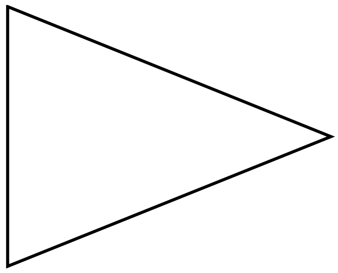

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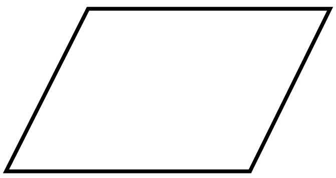

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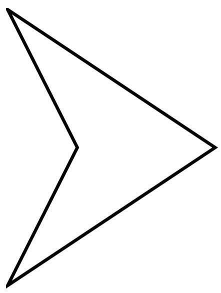

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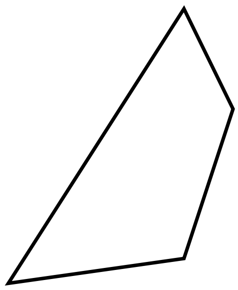

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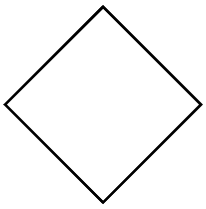

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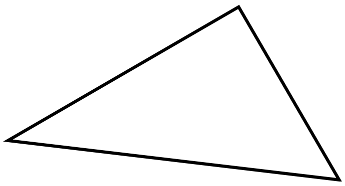

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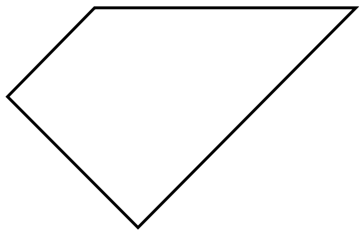
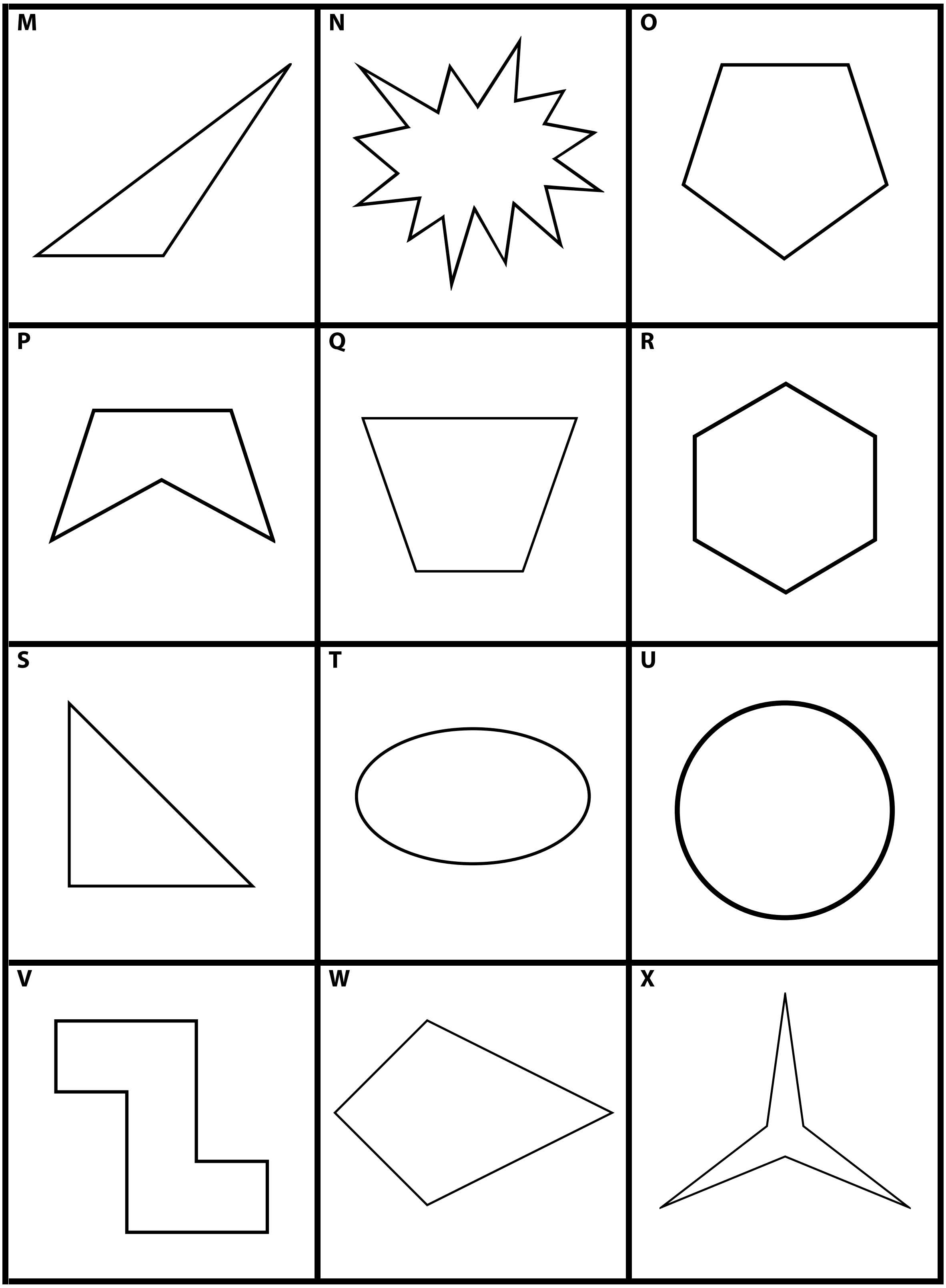


SS1 Card set A – *Shapes* (page 2)



SS1 Sheet 1 – *Classifying by symmetry*

	No rotational symmetry	Rotational symmetry
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		

SS1 Sheet 2 – *Classifying by regularity*

	Regular	Irregular
Triangle		
Quadrilateral		
Pentagon		
Hexagon		

SS1 Sheet 3 – *Classifying triangles*

	No right angles	One right angle
No sides equal		
Two sides equal		
Three sides equal		

SS1 Sheet 4 – *Classifying quadrilaterals*

	No parallel sides	Two parallel sides	Two pairs of parallel sides
No equal sides			
Two equal sides			
Two pairs of equal sides			

SS1 Sheet 5 – *Classifying by area and perimeter*

	Small area	Large area
Small perimeter		
Large perimeter		

SS2 • Understanding perimeter and area

Mathematical goals

To help learners to:

- understand the difference between perimeter and area.

To give learners practice in:

- calculating the area of rectangular shapes;
- calculating the perimeters of rectangular shapes.

Starting points

Learners may have had some previous experience of calculating perimeter and area.

Materials required

Possibly:

- a large block of chocolate.

For each learner you will need:

- several sheets of centimetre squared paper;
- pencil and ruler;
- mini-whiteboard.

Time needed

At least 30 minutes.

Suggested approach

Beginning the session

Show the group a large bar of chocolate (real if possible; otherwise use a diagram) that has many squares, e.g. 36. Assume that all the squares are 1 cm by 1 cm. Ask learners to explain what is meant by the area and the perimeter of the bar, rather than simply state the rule for calculating them. Discuss the difference between area (i.e. in this case, the area is the number of squares expressed in cm^2) and perimeter (i.e. the distance all the way around the outside).

Working in pairs (1)

Ask learners to work in pairs and give each pair some sheets of squared paper, a pencil and a ruler. Explain that they have to rearrange the squares to make different rectangles but they must keep the same number of squares. Their task is to find the arrangement that makes the rectangle with the longest perimeter and the arrangement that makes the shortest perimeter. They should draw the rectangles on their squared paper to help their discussion.

Whole group discussion (1)

Share answers and identify which arrangement has the longest perimeter and which has the shortest. Explain that they can now divide the squares into smaller areas, i.e. the length and width of the rectangle, measured in centimetres, no longer have to be integers. Ask for suggestions and check, as a group, to see if the perimeter can be made bigger or smaller.

If an interactive whiteboard is available, the length of the rectangle can be put onto a spreadsheet and the width and perimeter calculated using the spreadsheet. This allows an exploration of a wider range of decimal numbers for the length (in particular, very small ones) which show the width getting bigger and the perimeter getting very big.

	A	B	C
1	Length	Width	Perimeter
2		=36/A2	=2*(A2+B2)

Copy the formulae downwards into the B and C columns, e.g.

	A	B	C
1	Length	Width	Perimeter
2	4.5	8	25

Working in pairs (2)

Change the specification to a fixed perimeter of, say, 40 cm and a variable area. Challenge learners to find the maximum and minimum areas.

Ask learners, using squared paper, to find possible rectangles with perimeter 40 cm, giving their lengths and widths.

Learners tend to find it much more difficult to fix the perimeter and change the area. They often draw rectangles that have an area of 40 cm^2.

Whole group discussion (2)

Share all the findings. If possible, use a spreadsheet to explore further with non-integers for the length and width.

	A	B	C
1	Length	Width	Area
2		=20-A2	=A2*B2

Copy the formulae downwards into the B and C columns, e.g.

	A	B	C
1	Length	Width	Area
2	4.5	15.5	69.75

Reviewing and extending the learning

- Use mini-whiteboards and ask learners to draw rectangles (with dimensions) that have:

  an area of 50 cm^2;
  or
  a perimeter of 12 cm;
  plus other examples of your own.

- Encourage learners to use compound rectangular shapes. Fix the perimeter and allow learners to explore which shapes give the largest area and which give the smallest.
- Repeat for a fixed area.

What learners might do next

Investigate areas and perimeters of other shapes.

SS3 • Dissecting a square

Mathematical goals

To enable learners to:

- express a part/whole diagram in fractions or percentages;
- convert a fraction to a percentage (using a calculator);
- calculate areas of rectangles, triangles, circles and parts of circles;
- add, subtract and multiply fractions.

These goals can be adapted for learners working at lower levels. For example, you may restrict the range of shapes to just rectangles.

Starting points

This session builds on learners' prior knowledge about fractions and percentages and about finding areas of rectangles, triangles and circles. Learners begin by completing a short discussion task that is designed to reveal their existing understandings and misunderstandings about fractions.

Materials required

For each small group of learners you will need:

- Sheet 1 – *Dissecting a square*;
- Sheet 2 – *Making up your own dissection* (two copies);
- Sheet 3 – *Further examples for discussion*;
- calculator;
- pencil, ruler and compasses.

For learners aiming at lower levels, just use Sheets 1 and 2.

The whole group discussions will be much easier if you make OHTs of some of the sheets. Alternatively, you can use an interactive whiteboard or data projector.

Time needed

This session will take from 1 to 2 hours.

Suggested approach

Beginning the session

Draw the following diagrams on the board. Explain that they show part of a road sign indicating a severe bend.

$$\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{4}{48}$$

So $\frac{4}{48}$ of the sign will be white.

Ask learners to tell you what has gone wrong with the calculation. It is often helpful to ask them to explain their thinking to their neighbours before answering.

Now discuss the following mistake in a similar way:

> In a supermarket you see an item marked '20% off'.
> You decide to buy two. The person behind the checkout says:
> "20% off each item
> That will be 40% off the total price."

Working in groups

Ask learners to work in pairs. Hand out copies of Sheet 1 – *Dissecting a square*. Ask learners to calculate the fraction and percentage of the whole square that each piece represents.

If you think some learners will find this dissection too difficult, use Sheet 2 – *Making up your own dissection* to prepare an alternative using rectangular pieces.

You may need to remind learners how to convert a fraction to a percentage using a calculator by dividing the numerator by the denominator and multiplying by 100. They can then use the percentages to check their answers.

The purpose of this activity is to encourage reasoning. Thus learners may have some difficulty when writing piece E on Sheet 1 as a fraction, but they may reason, for example, that E can be calculated by subtracting a small triangle that is $\frac{1}{16}$ of the whole square from a right angled triangle that is $\frac{1}{8}$.

Look out for learners who make the same mistakes that were in the introductory discussion. For example, they might answer that piece A is $\frac{1}{4}$ but piece C is $\frac{3}{4}$, thus changing their view of the size of one unit.

Whole group discussion

Discuss different strategies for solving the problem. For example, for piece B:

> Divide the whole square into small squares like B and show that 16 Bs make up the whole. Therefore B = $\frac{1}{16}$.

or

> B has a length of $\frac{1}{4}$ and width of $\frac{1}{4}$ so altogether it has an area of $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.

Similarly, ask learners to share different approaches for finding piece E.

> $D + E + F = \frac{1}{2} = \frac{8}{16}$;
>
> $D = \frac{5}{16}$;
>
> $F = \frac{1}{8} = \frac{2}{16}$;
>
> So E must be $\frac{1}{16}$.

or

> E has a base of $\frac{1}{4}$ and a height of $\frac{1}{2}$.
>
> Using area of triangle formula: $E = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{16}$.

Discuss how fractions can be converted to percentages with a calculator and that these percentages can be added to check that they make 100.

Working in groups

Now hand out two copies of Sheet 2 – *Making up your own dissection* to each pair of learners. Encourage each learner to create a problem similar to Sheet 1 for a partner to solve. They should try to make the problem quite challenging for the partner, but they must supply the correct answers on the back.

Learners aiming at lower levels may be asked to restrict their dissections to rectangles, and then triangles.

Learners may use a variety of shapes, e.g. squares, rectangles, triangles or circles. They should try to create problems according to their own level of confidence and competence.

A range of examples is given on Sheet 3 – *Further examples for discussion*.

Reviewing and extending learning

Finally, hold a whole group discussion sharing some of the problems produced by learners or from Sheet 3 – *Further examples for discussion*.

This may be facilitated using OHTs. These could be made by asking learners to trace some of their problems onto a blank acetate, or by photocopying the examples in Sheet 3 onto acetate.

What learners might do next

Learners could be given particular fraction calculations to illustrate. For example:

Draw me a dissection to show that:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1; \frac{1}{5}+\frac{1}{4}+\frac{11}{20}=1.$$

Draw me a dissection made entirely of triangles to show:

$$\frac{1}{2}+\frac{2}{5}+\frac{1}{10}=1.$$

Further ideas

This session invites learners to create and solve their own problems. This type of activity may be used throughout the mathematics curriculum. Examples in this pack include:

A2 Creating and solving equations;

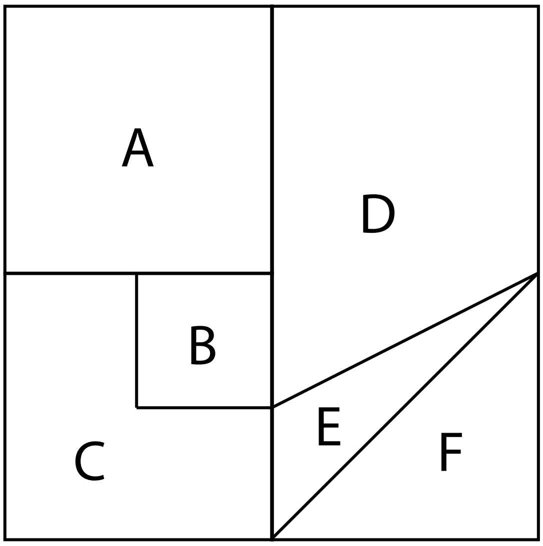
N10 Developing an exam question: number;

A8 Developing an exam question: generalising patterns;

SS8 Developing an exam question: transformations;

S7 Developing an exam question: probability.

SS3 Sheet 1 – *Dissecting a square*



Piece A is $\frac{1}{4}$ (or 25%) of the whole square.

1. What fraction and percentage of the whole square are the other pieces? Explain how you know.

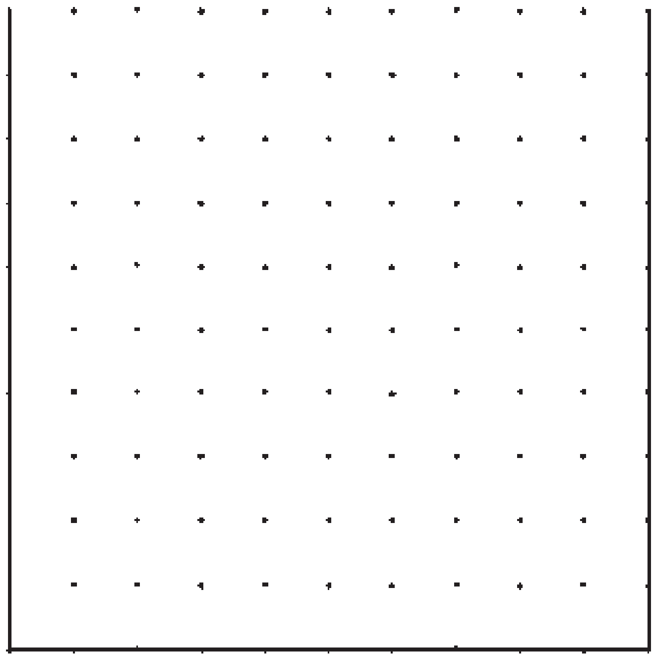
Piece	Fraction	Percentage	Reason
A			
B			
C			
D			
E			
F			

2. Check that your answers add up to a whole unit (or 100%).

SS3 Sheet 2 – *Making up your own dissection*

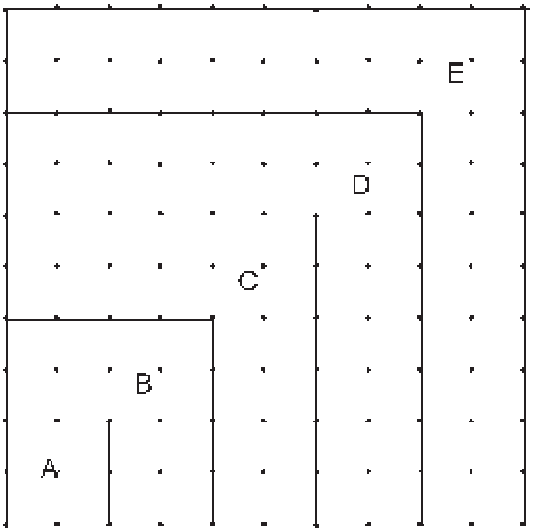
Divide this square into fewer than ten pieces and label them A, B etc. Try to make pieces with different sizes and shapes. On the back of this sheet, write down the size of each piece using fractions and percentages.

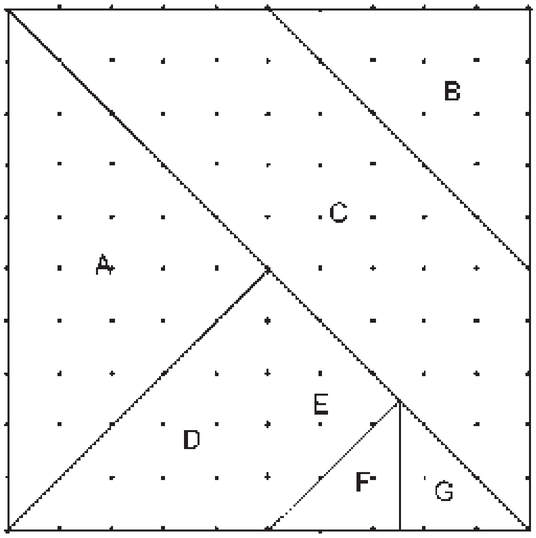
Give this sheet to your partner and see if they get the same answers.

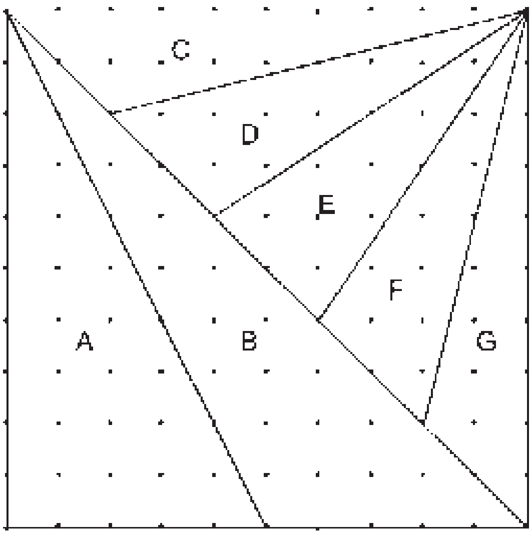


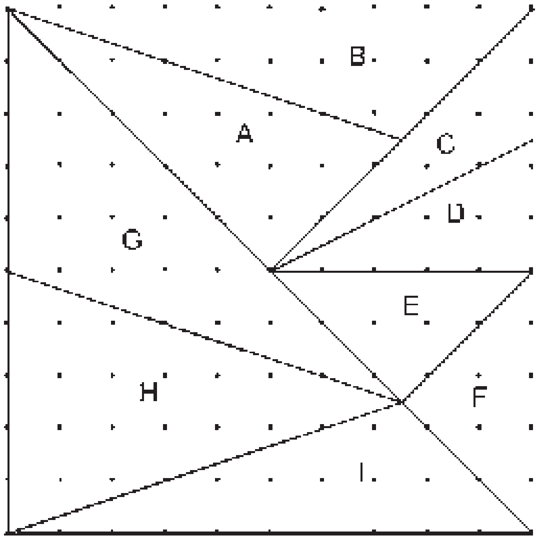
Piece	Fraction	Percentage	Reason
A			
B			
C			
D			
E			
F			

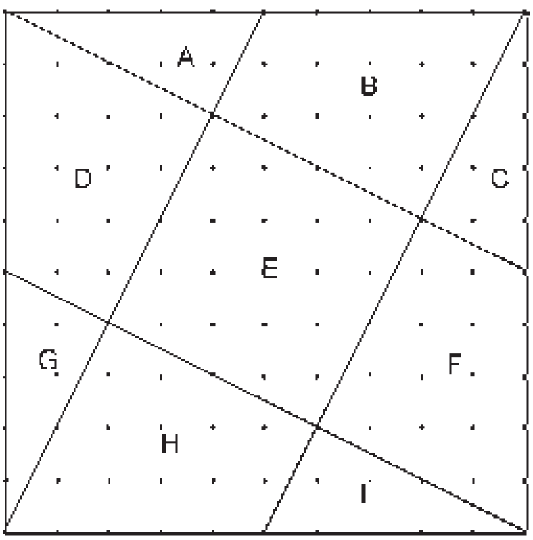
SS3 Sheet 3 – *Further examples for discussion*

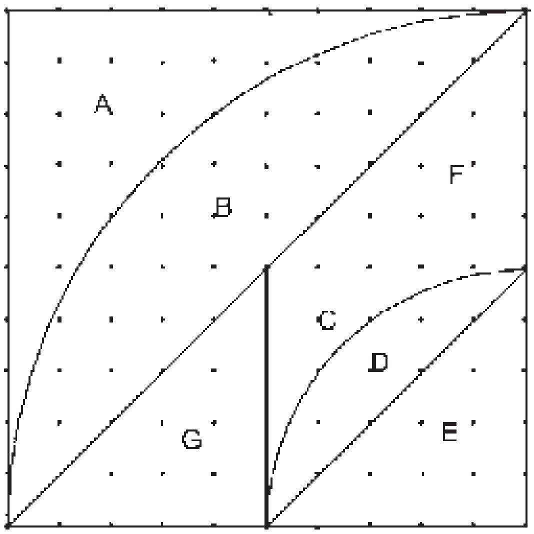












SS4 • Evaluating statements about length and area

Mathematical goals

To help learners to:

- understand concepts of length and area in more depth;
- revise the names of plane shapes;
- develop reasoning through considering areas of plane compound shapes;
- construct their own examples and counter-examples to help justify or refute conjectures.

Starting points

Learners should have already encountered the concepts of area and perimeter before beginning this activity. This activity will consolidate their understanding and overcome some common misconceptions, such as the notion that perimeter and area are in some way interrelated.

Learners are given a number of mathematical statements on cards. They have to try to justify or refute these statements using their own examples and arguments. They create posters showing their collaborative reasoning.

Materials required

For each small group of learners you will need:

- Card set A – *Statements*;
- Card set B – *Hints*;
- blank A4 sheets on which to make mini-posters;
- glue stick.

Time needed

This will depend on the number of *Statements* cards you give to each learner. One hour is probably the minimum time needed.

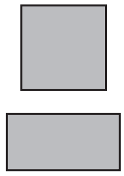
Suggested approach

Beginning the session

This activity is best introduced with a whole group discussion.

Choose one of the statements from Card set A and write it on the board:

> If a square and a rectangle have the same area,
> the square has the smaller perimeter.



Demonstrate the process that learners should adopt when tackling this activity, by working through this example together.

Step 1. Understand the problem

Explain that the learners have to decide whether the statement is always true, sometimes true, or never true. What does this mean? Well, we need to decide whether the statement is true or not for all possible squares and rectangles that have the same area.

What does the word 'area' mean? How do we calculate it?
What does the word 'perimeter' mean? How do we calculate it?

Step 2. Try some examples

Give me some dimensions for a square.	(6 cm × 6 cm)
What is the area of that square?	(36 cm^2)
What is its perimeter?	(24 cm)
Now give me some dimensions for a rectangle with the same area as the square.	(12 cm × 3 cm)
What is the perimeter of the rectangle?	(30 cm)
So is the statement true for this example?	(Yes)
Can you give me a different rectangle with the same area?	(9 cm × 4 cm)
What is the perimeter of the rectangle?	(26 cm)
So is the statement true for this example?	(Yes)

In this way, help learners to generate examples and then to offer conjectures.

Step 3. Make conjectures

Learners may begin to notice, for example, that, for a given area, as the dimensions of the rectangle become more nearly equal, so the perimeter reduces. This might lead to the conjecture that the square has the smallest perimeter for a given area. Thus the statement appears to be always true.

Step 4. Try to disprove or justify the conjectures

Can we see why the statement must be true? This statement might be too difficult for learners to prove algebraically, but they may be able to test particular cases in an organised way. So, when the area of the rectangle is 36 cm^2, possible perimeters are:

One side of the rectangle	1	2	3	4	5	**6**	7	8	9	10	11	12
Other side (to 1 d.p.)	36	18	12	9	7.2	**6**	5.1	4.5	4	3.6	3.3	3
Perimeter (to 1 d.p.)	74	40	30	26	24.4	**24**	24.2	25	26	27.2	28.6	30

This table (which might be supported by a graph) suggests that the statement is always true, except when the rectangle takes the same shape as the square (i.e. 6 × 6) when of course their areas are equal.

Working in groups

Ask learners to work in pairs. Give Card set A – *Statements* to each pair and ask them to choose a statement to work on.

Learners who you think may struggle should be guided towards one of the earlier statements.

Learners should try to decide whether their chosen statement is always, sometimes or never true, using the process described above. Remind them of the need to test specific examples in an organised way. In many cases they may find it possible to prove their ideas using reasoning that doesn't depend on numbers at all.

Learners should choose a statement, stick it in the middle of a sheet of A4 paper and write their reasoning around it. If a different A4 poster is made for each statement, these can be gathered and displayed for later discussion and critical comment.

If a pair of learners get stuck, offer the appropriate card from Set B – *Hints*.

When two pairs have completed the same problem, ask them to exchange posters and comment on the reasoning of the other group.

Reviewing and extending learning

Invite pairs of learners to describe one problem that they think they have solved successfully and the reasoning that they employed. Then ask other pairs who have solved the same problem to show their reasoning. Ask the remaining learners to say which reasoning they found most clear and convincing and why.

What learners might do next

Session **SS5 Evaluating statements about enlargement** is a good follow-up to this session. It develops the ideas of perimeter and area and includes some discussion of 3D solids.

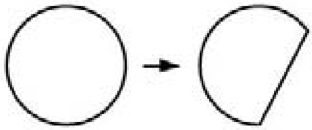
Further ideas

This activity is about examining a mathematical statement and deciding on its truth or falsehood. This idea may be used in many other topics and levels. Examples in the pack include:

- **N2** **Evaluating statements about number operations;**
- **A4** **Evaluating algebraic expressions;**
- **S2** **Evaluating probability statements.**

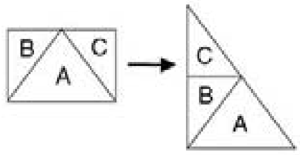
SS4 Card set A – *Statements*

A



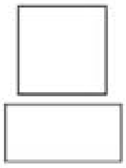
When you cut a piece off a shape you reduce its area and perimeter.

B



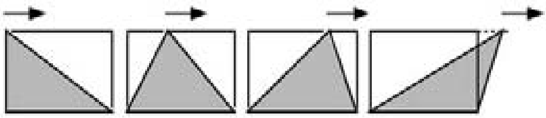
When you cut a shape and rearrange the pieces, the area and perimeter stay the same.

C



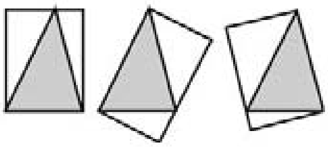
If a square and a rectangle have the same perimeter, the square has the smaller area.

D



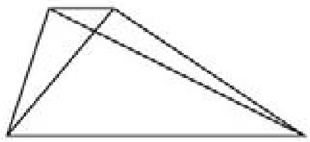
Slide the top corner of a triangle from left to right. The area of the triangle stays the same.

E



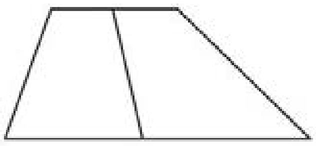
Draw a triangle. There are three ways of drawing a rectangle so that it passes through all three vertices and shares an edge with the triangle. The areas of the three rectangles are equal.

F



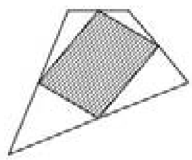
Draw a trapezium and draw its diagonals. The shape is now split into four triangles. Exactly two of these triangles are equal in area.

G



If you join the mid points of the opposite sides of a trapezium, you split the trapezium into two equal areas.

H

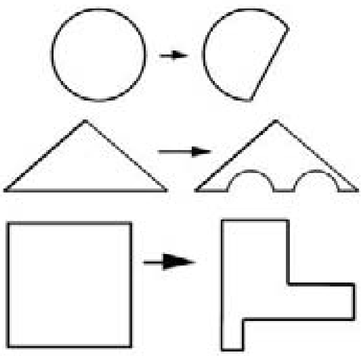


If you join the mid points of the sides of a quadrilateral, you get a parallelogram with one half the area of the original quadrilateral.

SS4 Card set B – *Hints*

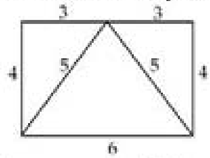
A

What happens to the area and perimeter with these cuts?



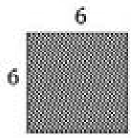
B

Draw a 6 cm by 4 cm rectangle and cut it into three pieces.



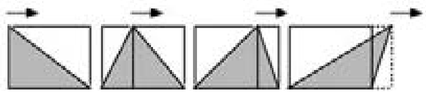
What are the area and the perimeter?
Use all three pieces to make each of the following shapes: *Rhombus, Trapezium, Parallelogram, Pentagon, Kite, Hexagon.*
Write down the area and the perimeter of each shape.
What areas and perimeters are possible?

C



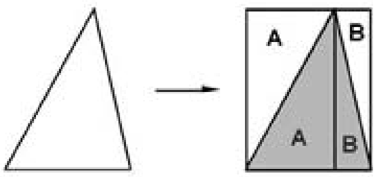
Draw some rectangles with the same perimeter as this square.
What is the area in each case?
Does the square have the smaller area?
Will this be true when you start with different squares?

D



Divide the shaded triangle into smaller triangles as shown.
What fraction of the rectangle is the shaded area in each case?
Will this work in extreme cases?

E



What fraction of each rectangle is the triangle?
What happens when the triangle contains an obtuse angle?

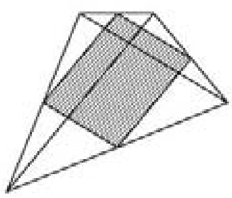
F

Begin by looking for triangles with a common base and the same height.
Try this for different trapezia.
Does it always work?
Can you get more than two equal areas? When?

G

Divide the diagram into triangles.
Can you see why some have the same base and height?
Try this for different trapezia.
Does it always work?
Can you get more than two equal areas? When?

H



Draw the diagonals of the quadrilateral.

SS5 • Evaluating statements about enlargement

Mathematical goals

To enable learners to:

- explore the relationship between linear and area enlargement;
- substitute into algebraic statements;
- discuss some common misconceptions about enlargement.

Starting points

Most learners will have covered some aspects of enlarging shapes before, but they may not have explored the relationships between linear, area and volume enlargement. In this session learners sort statements into categories: always, sometimes or never true. Throughout their work on this activity, learners justify and explain their decisions using examples and counter-examples. They present their ideas to the rest of the group by means of a poster.

Materials required

- OHT of Sheet 1 – *Perimeters and areas*.

For each small group of learners you will need:

- Sheet 1 – *Perimeters and areas*;
- Card set A – *Doubling statements*;
- Card set B – *Diagrams*;
- large sheet of paper for making a poster;
- felt tip pen;
- glue stick;

and possibly:

- Sheet 2 – *Rep-tiles: enlarging different shapes*.

Time needed

At least 1 hour.

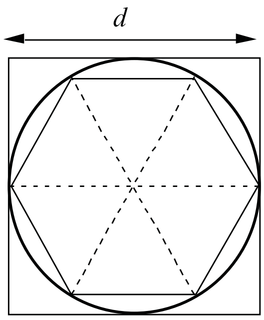
Suggested approach

Beginning the session

Ask learners, working alone, to complete Sheet 1 – *Perimeters and areas*. This helps them to review the meaning of area and perimeter and reminds them (with some justification) of the formulae πd and πr^2.

Whole group discussion

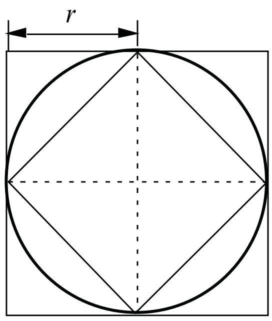
After a short while, hold a whole group discussion to review this work, using the OHT of Sheet 1 – *Perimeters and areas*.



Show the first drawing.

What is the perimeter of the square? ($4d$)
What is the perimeter of the hexagon? ($3d$)
The perimeter of the circle (or circumference) lies between $3d$ and $4d$. Does anyone know what it is?

The value is about $3.142 \times d$. The number 3.142 is only approximate. It is impossible to write the number down exactly, as it has an infinite number of decimal places with no pattern to them. We often use the letter π to represent this number.



Now show the second drawing.

What is the area of the large square? ($2r \times 2r = 4r^2$)
What is the area of the small square? (Half as much = $2r^2$)
The area of the circle lies between $2r^2$ and $4r^2$

The area of the circle is about $3.142 \times r^2$ or πr^2, where π turns out to be the same number as before.

Working in groups

Organise learners into groups of two or three.

Give each group Card set A – *Doubling statements*, a large sheet of paper, a glue stick and a felt tip pen. Ask learners to divide their sheet into three columns and head the columns with the words 'Always true', 'Sometimes true', 'Never true'.

Talk through the following steps with learners, maybe using one or two examples from the sheet.

1. Choose a pair of statements

Choose a pair of statements about a particular shape. If you are confident, choose a harder shape; if you are less confident, choose the rectangle.

2. Look at a special case

   Draw the shape you have chosen and give it some measurements. Draw a larger version with double the dimensions. Calculate areas and perimeters. Do the statements appear to be true?

3. If you think a statement is true

   Try to show why the statement works by drawing or by using algebra. Let the lengths on the drawing be x and y for example, and calculate the other lengths in terms of these.

4. If you think a statement is false

   Try to say how the statement should be changed so that it becomes true. Explain why you are sure that your version is true.

Learners should take it in turns to place a card in one of the columns and justify this response to their partner. Their partner must challenge them if the explanation has not been clear and complete. When the pair agrees, they should stick the card down and write examples and explanations to show the reasoning behind the decision.

It is not necessary for all learners to tackle every statement. Encourage groups to begin with statements at an appropriate level of difficulty.

Explain that learners will be expected to present their conclusions to the rest of the group, using their posters.

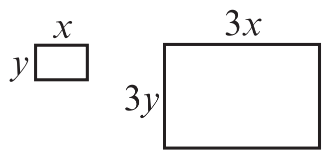
If learners get stuck, you may like to offer them an appropriate card from Card set B – *Diagrams*. These provide some helpful hints and constructions that will get them started.

When learners make rapid progress, encourage them to explore additional shapes and also explore what happens when the dimensions are trebled.

Reviewing and extending learning

Ask each pair of learners to present what they have found to the rest of the group, using their posters. As they do this, ask other learners for further examples and reasons. Encourage numerical, geometrical and algebraic reasoning.

For example, if someone has considered trebling the sides of a rectangle, one could compare the perimeters $2x + 2y$ and $6x + 6y$, and see that, whatever values x and y take, one perimeter will be three times as long as the other. Similarly, it is immediately clear that the area has increased by a factor of 9.



What learners might do next

Learners may like to look at the enlargement of other shapes. Sheet 2 – *Rep-tiles: enlarging different shapes* has been provided for this purpose.

Further ideas

This activity is about examining a mathematical statement and deciding on its truth or falsehood. This idea may be used in many other topics and levels. Examples in this pack include:

A4 Evaluating algebraic expressions;

SS4 Evaluating statements about length and area;

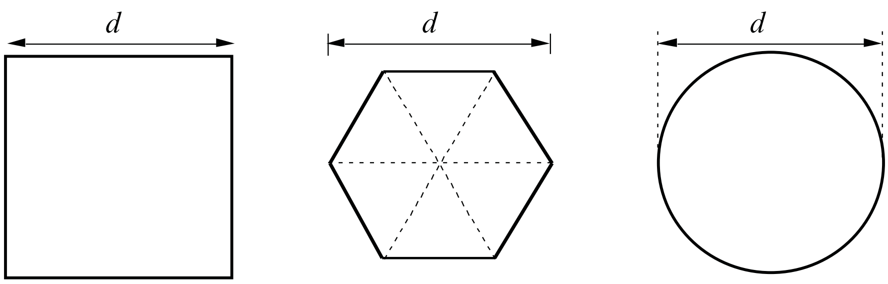
S2 Evaluating probability statements.

SS5 Sheet 1 – *Perimeters and areas*

How would you explain the terms 'perimeter' and 'area' to someone who has never heard of them?

Think about this question for a few minutes and note down your thoughts. Then tackle the questions below.

Perimeter



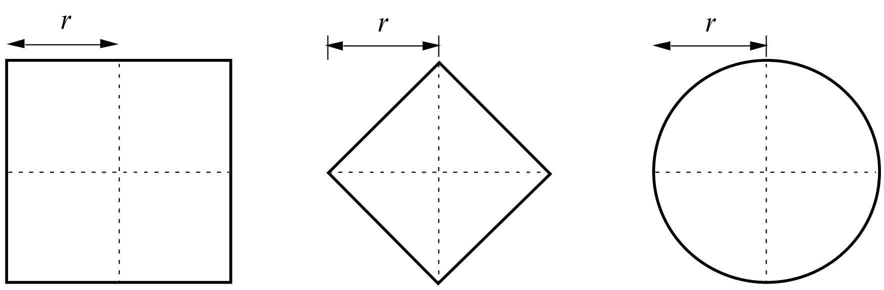
Which of these shapes has the greatest perimeter?

Which has the smallest perimeter?

Can you write each perimeter in terms of the letter d?

If you can't do this exactly, make an estimate.

Area



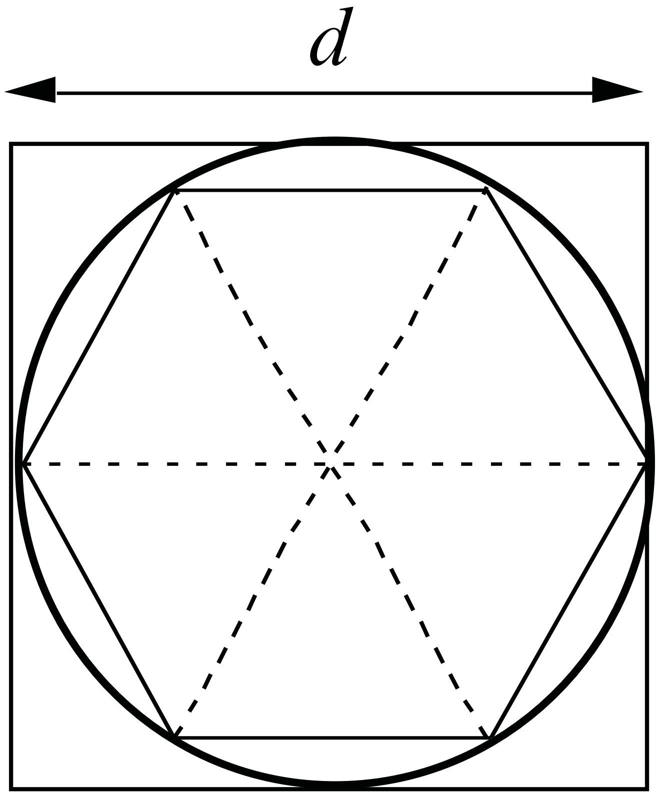
Which of these shapes has the greatest area?

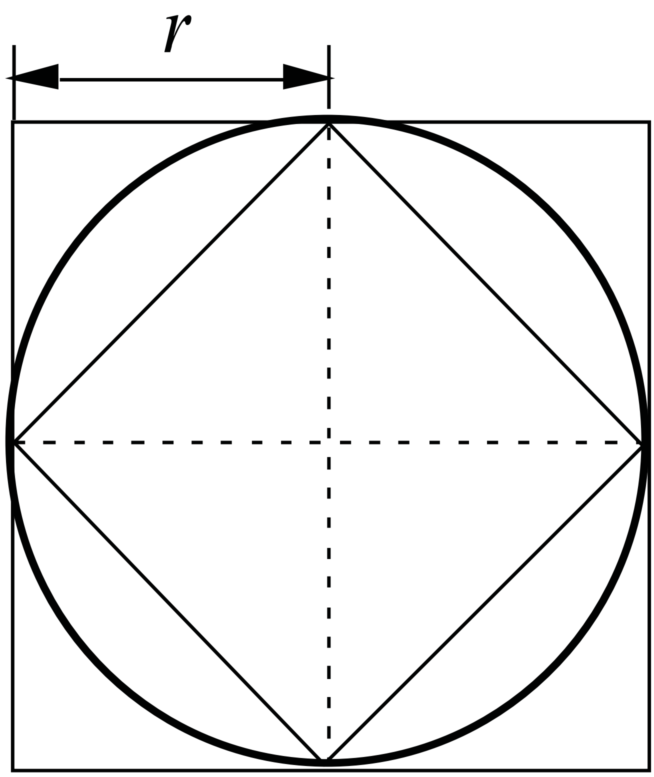
Which has the smallest area?

Can you write each area in terms of the length r?

If you can't do this exactly, make an estimate.

SS5 OHT – *Perimeters and areas*

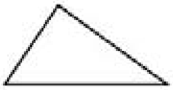




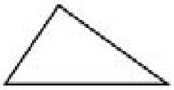
SS5 Card set A – *Doubling statements*

If you double the lengths of the sides of a rectangle, its perimeter doubles.

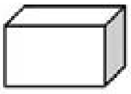
If you double the lengths of the sides of a rectangle, its area doubles.



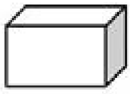
If you double the lengths of the sides of a triangle, its perimeter doubles.



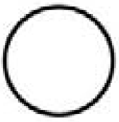
If you double the lengths of the sides of a triangle, its area doubles.



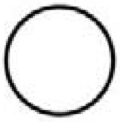
If you double the lengths of the edges of a cuboid, its volume doubles.



If you double the lengths of the edges of a cuboid, its surface area doubles.



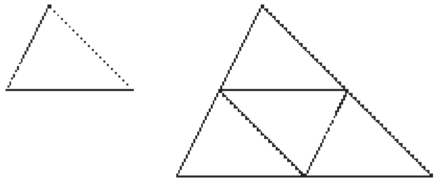
If you double the radius of a circle, its area doubles.



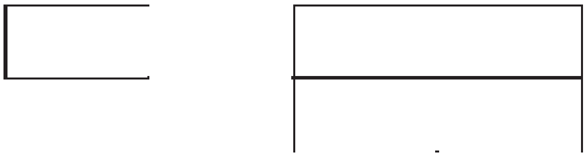
If you double the radius of a circle, its circumference doubles.

SS5 Card set B – *Diagrams*

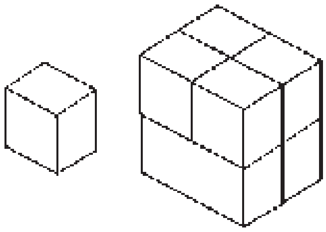
Triangles



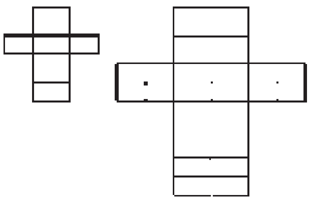
Rectangles



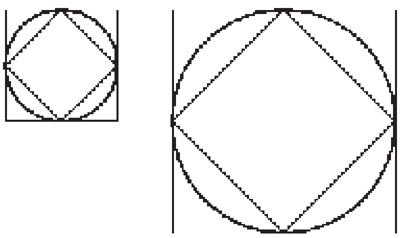
Cuboid volumes



Cuboid surface areas



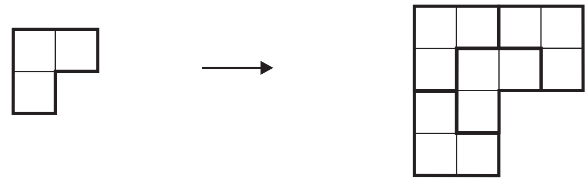
Circles

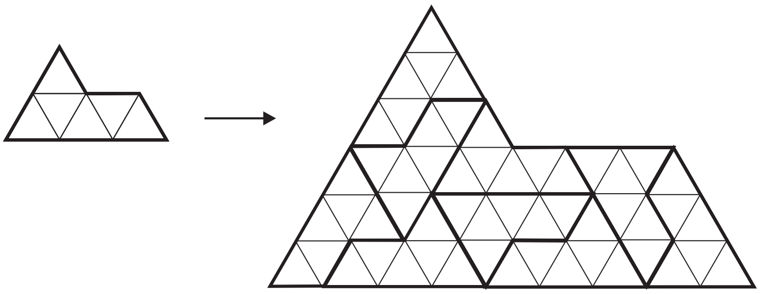


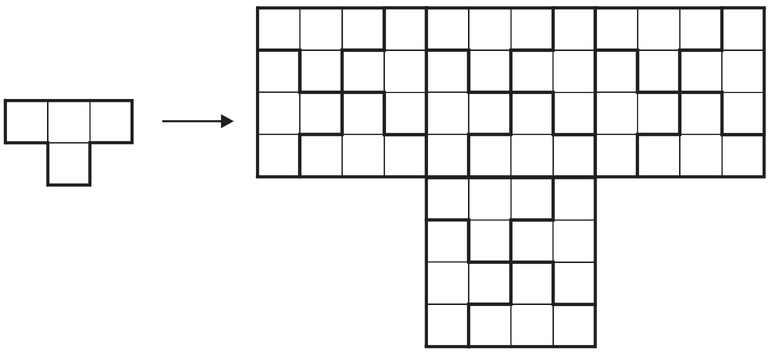
SS5 Sheet 2 – *Rep-tiles: enlarging different shapes*

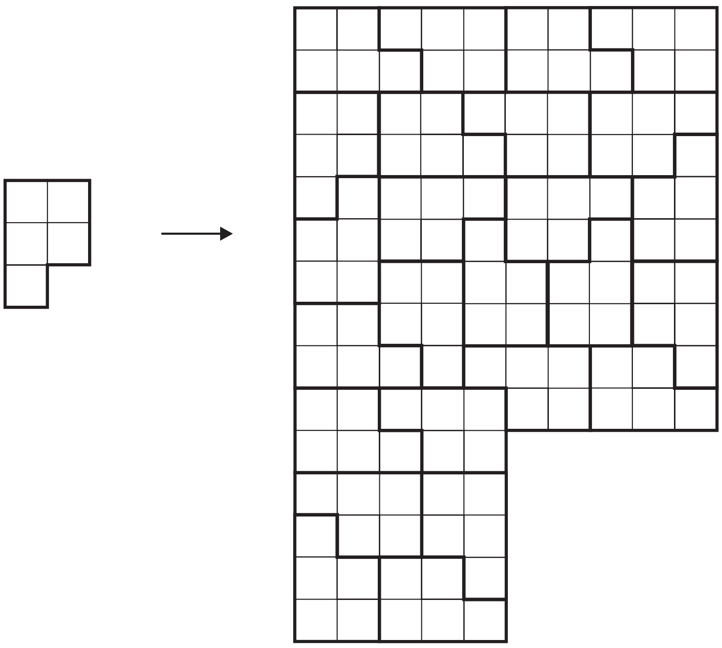
Each small tile can be tessellated to give a larger version of itself.

Consider how the lengths, perimeters and areas of the tiles change as they are enlarged and write about the relationships you find.









SS6 • Representing 3D shapes

Mathematical goals

To enable learners to:

- interpret 2D representations of 3D shapes;
- analyse 3D shapes using plans, elevations, and isometric drawings;
- develop their reasoning ability in spatial contexts.

Starting points

Many learners may have met these ideas before, but they may still experience difficulties in this area. In this session, learners match cards showing 3D drawings of 'cube houses' with plans and elevations. They are then provided with an opportunity to create their own cube houses and draw their plans, elevations and isometric representations.

This session is much enhanced if the accompanying software *Building houses* is used. This is a computer program produced by the Freudenthal Institute for secondary education in Holland. It is available on the DVD-ROM and, along with many other useful programs, is on the website www.fi.uu.nl/

Materials required

- OHT 1 – *Plans and elevations*;
- OHT 2 – *Isometric drawing*.

For each small group of learners you will need:

- Card set A – *Perspectives*;
- Card set B – *Plans and elevations*;
- Card set C – *Isometric drawings*;
- squared and triangular grid and isometric graph paper;
- a supply of interlocking cubes (optional, but very helpful).

For learners aiming at lower level qualifications, a supply of interlocking cubes is particularly helpful.

During discussions with learners, it is helpful if you have overhead projector transparencies of the cards as well as OHTs 1 and 2.

Time needed

Learners usually need between 1 and 2 hours but this depends on how many of the problems in the software are used.

Suggested approach

Beginning the session

Provide each learner with a sheet of squared grid paper. Place OHT 1 – *Plans and elevations* on the projector, hiding the plans and elevations. Explain that what learners can see is a perspective drawing of a model of a 'cube house'. Ask learners to draw, on squared grid paper, what they think the house looks like from above (its plan), from the front looking in the direction of the arrow (its front elevation), and from the right hand side (its right side elevation). Collect and discuss their suggestions before revealing the plans and elevations on the OHT. There may be more than one correct answer.

Alternatively, use a data projector and the computer program *Building Houses 1* to do this (see page SS6-4 – *Using the computer software (1)*). This has the advantage that the building can be turned round gradually so that learners can see all sides.

Working in groups

Ask learners to sit in pairs (or threes) and give each group Card set A – *Perspectives* and Card set B – *Plans and elevations*.

Ask learners to take it in turns to match each *Perspective* card with the correct *Plan and elevation* card. As they do this, they should explain to their partner(s) why they have matched the cards in that way. When they have given their explanation, their partner(s) should either challenge what they have said or say why they agree.

When learners have done this, ask them to sketch, on *Perspective* card F, a cube house that matches the remaining three *Plan and elevation* cards. Alternatively, they might make a model of the missing cube house using interlocking cubes. Learners who find this easy may be challenged to try to find more than one solution.

When learners are comfortable with their final results, ask them to compare the positions of their cards with those of another group. Ask learners to reconcile any differences that emerge.

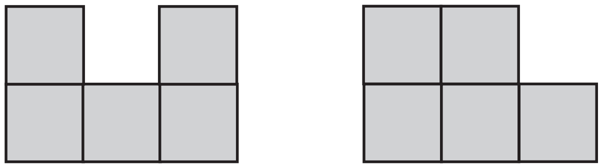
When a group has finished these activities, ask them to make their own cube house, using up to twenty interlocking cubes. They should then draw its plan and elevations on squared paper. These drawings should then be given to another group who should try to draw the 3D cube house.

You may also wish to introduce the convention of isometric drawings of 3D shapes by showing OHT 2 – *Isometric drawings*. This shows *Perspective* card F together with *Isometric* card L. Explain how this shows how cubes can be hidden in the isometric view.

Give out Card set C – *Isometric drawings*. Learners can try to match *Isometric* cards G–L with *Perspective* cards A–F.

Reviewing and extending learning

Follow up the learning by posing some open problems, using plans and elevations, e.g.



Here is a front and side elevation.

- Draw me a possible plan view.
- Draw me an impossible plan view.
- What is the greatest possible number of cubes that can be in the house?
- What is the least possible number?

Further ideas

This activity uses multiple representations to deepen learners' spatial sense. This type of activity may be used in any topic where a range of representations is used. Examples in this pack include:

A1 Interpreting algebraic expressions;

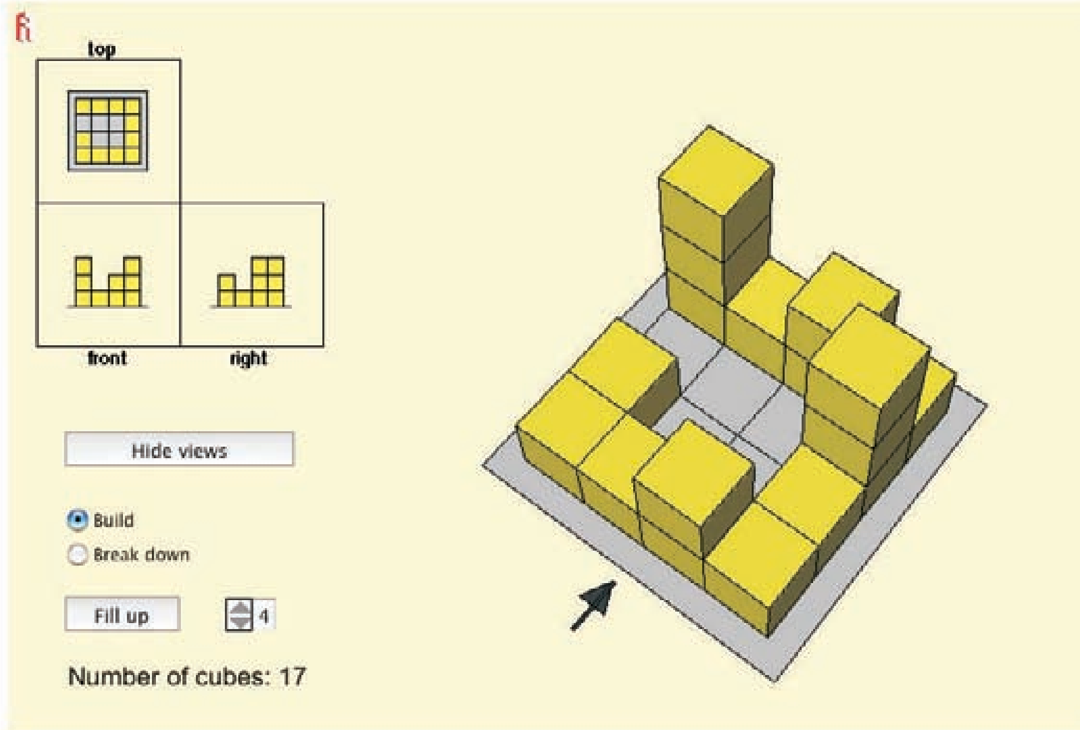
SS7 Transforming shapes;

S5 Interpreting bar charts, pie charts, box and whisker plots;

S6 Interpreting frequency graphs, cumulative frequency graphs, box and whisker plots.

Using the computer software (1)

Building houses 1



This piece of software, particularly if used with an interactive whiteboard, considerably enhances the introduction to this session. It also allows learners to check their card matching and provides a medium in which learners can construct their own examples.

The opening screen is set to build a 'house' on a square base of size 4. The size of the square base can be changed using the 'Up' and 'Down' buttons. The black arrow always shows the direction of the front of the house. You can click and drag the house around to obtain different views.

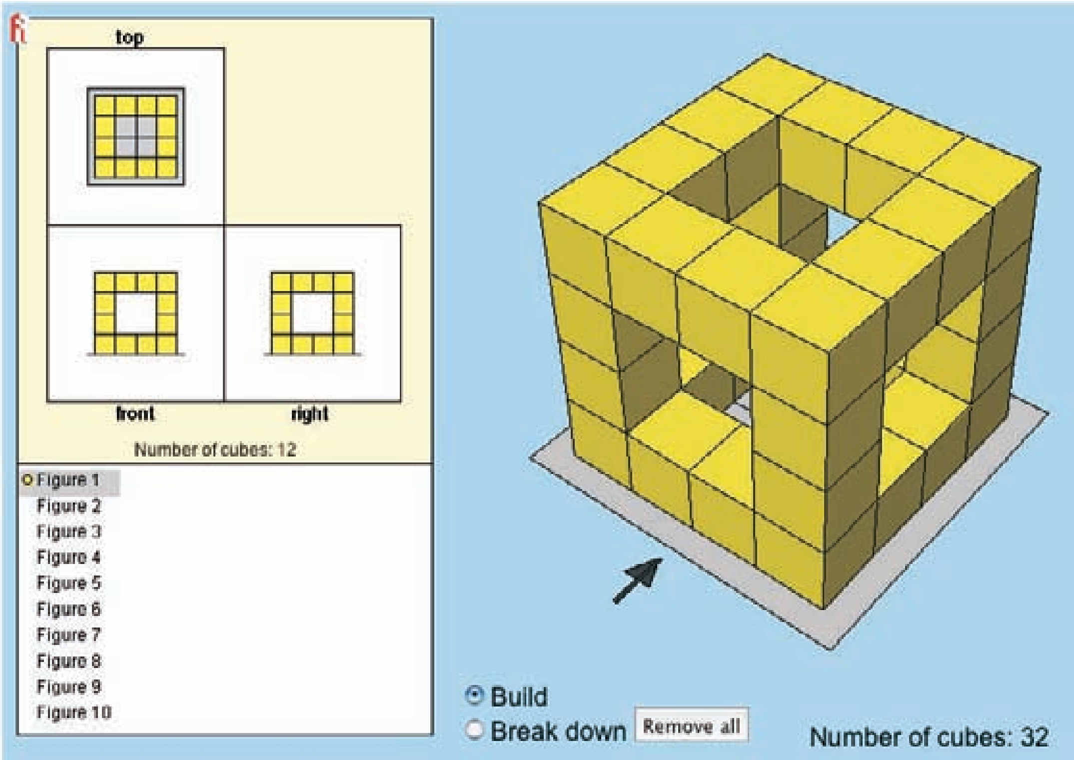
Clicking on the 'Fill up' button builds a cubical house 4 by 4 by 4: the number of cubes is 64. The plan, front and side elevations of the cube are shown in the top left of the screen. Clicking on the 'Remove all' button breaks down the house.

To build a more interesting house, click on any square on the base; a cube appears on the 3D representation, and on the plan and elevations. Click on the top (or side) of any cube to place another cube on top (or to the side) of the first cube. Clicking on the 'Break down' button, then clicking on a cube, deletes that cube.

Problems can be posed by hiding the plan and elevations using the 'Hide views' button, creating a house, then asking learners to predict what the plan and elevations will look like when they are revealed.

Using the computer software (2)

Building houses 2

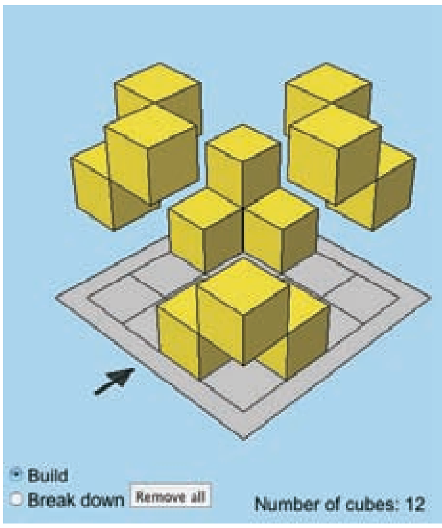


In this program, learners are challenged to construct 3D models from given plans and elevations.

The program provides the plans and elevations (see top left in the screen shown above) and learners have to build appropriate houses. For example, figure 1 has been successfully created in the screen shown above. When this happens, a green blob appears next to the figure number.

The program, however, presents a further challenge. In the above example, it suggests that only twelve cubes are needed to build the model.

This is possible using the software, though it involves floating cubes, as shown in the screen on the right. When learners succeed at this, a yellow blob appears next to the figure number.



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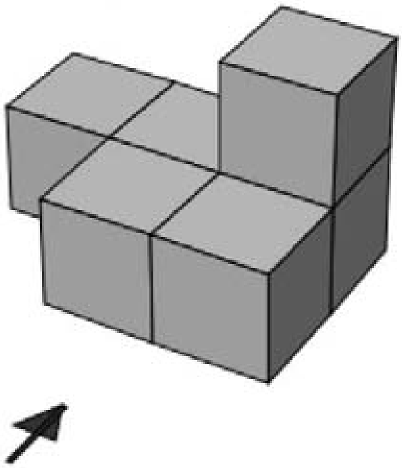
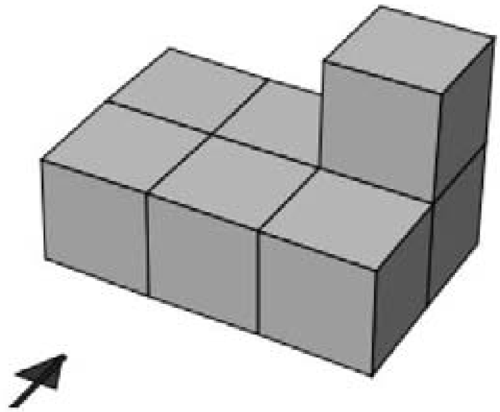
SS6 Card set A – *Perspectives*

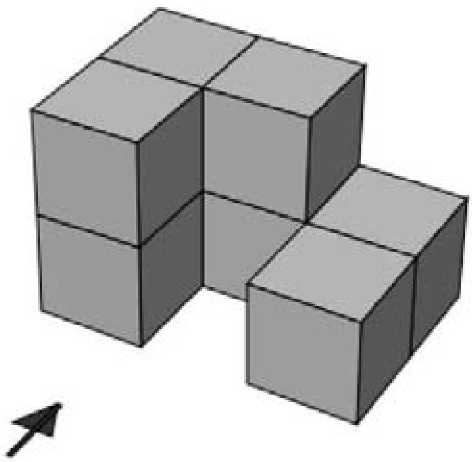
Perspective card A



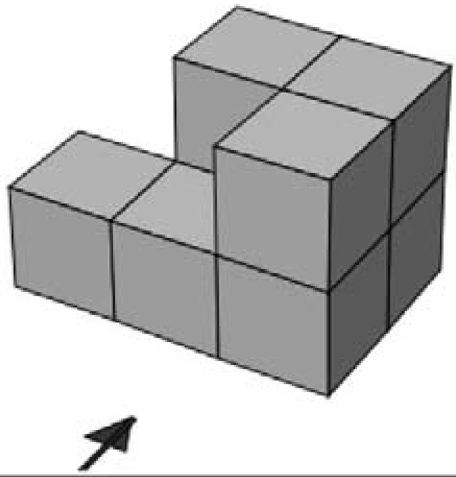
Perspective card B



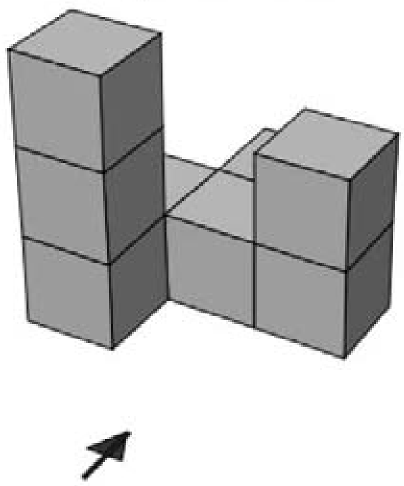
Perspective card C



Perspective card D



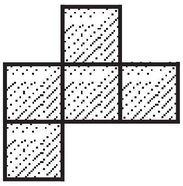
Perspective card E



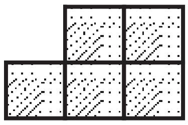
Perspective card F

SS6 Card set B – *Plans and elevations*

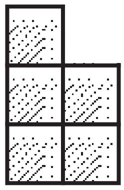
1 Plan card



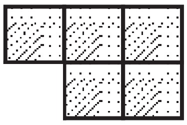
2 Front card



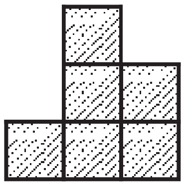
3 Right side card



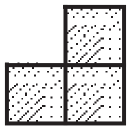
4 Plan card



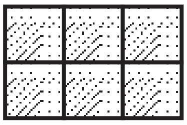
5 Front card



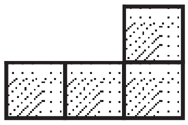
6 Right side card



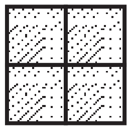
7 Plan card



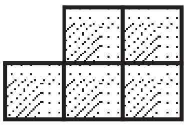
8 Front card



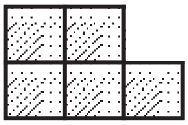
9 Right side card



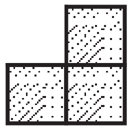
10 Plan card



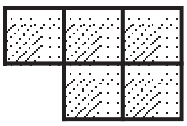
11 Front card



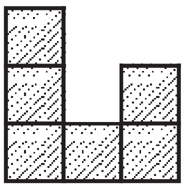
12 Right side card



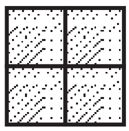
13 Plan card



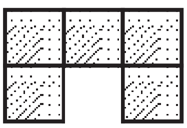
14 Front card



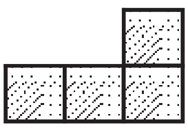
15 Right side card



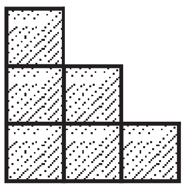
16 Plan card



17 Front card

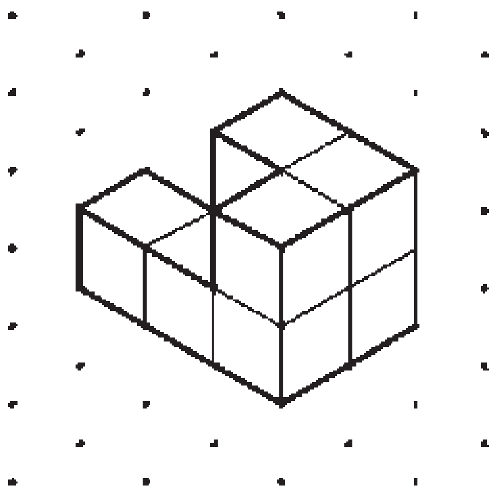


18 Right side card

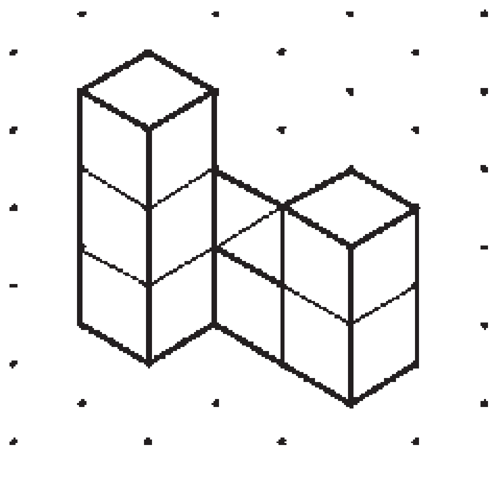


SS6 Card set C – *Isometric drawings*

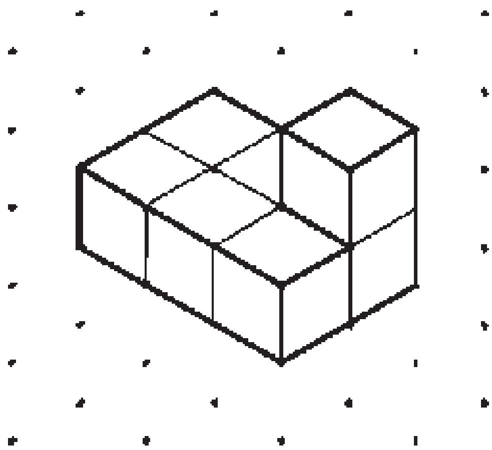
Isometric card G



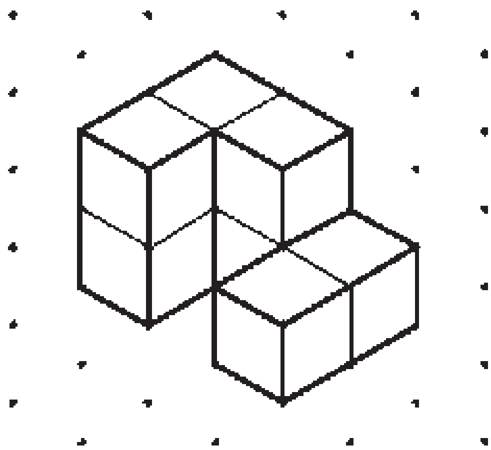
Isometric card H



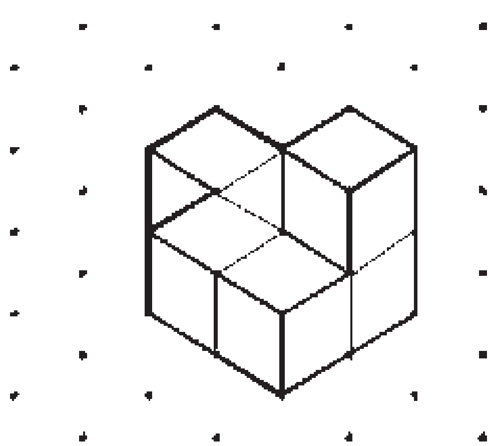
Isometric card I



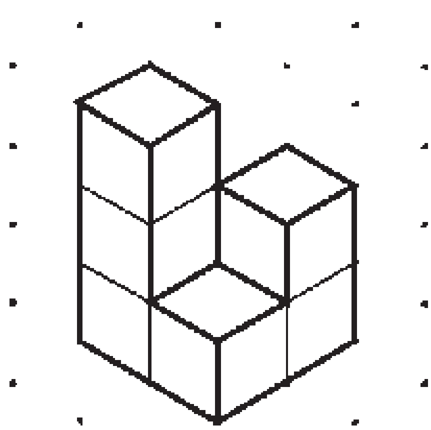
Isometric card J



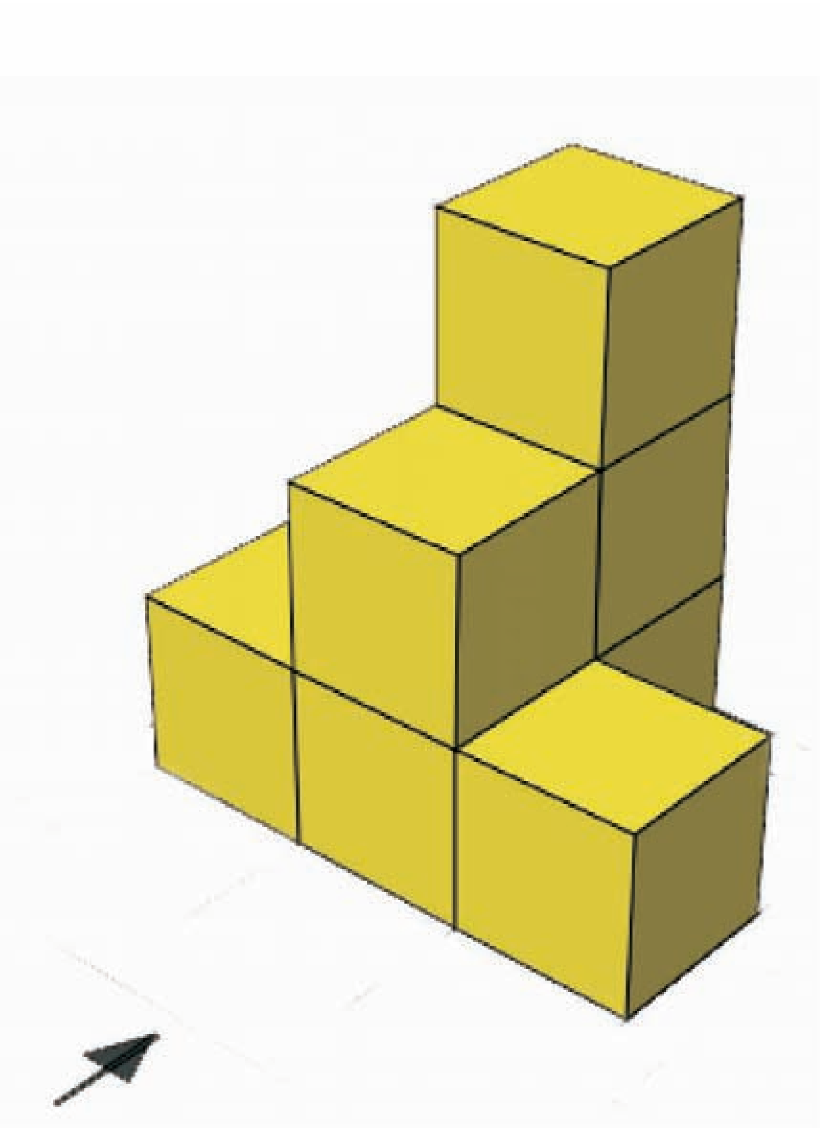
Isometric card K

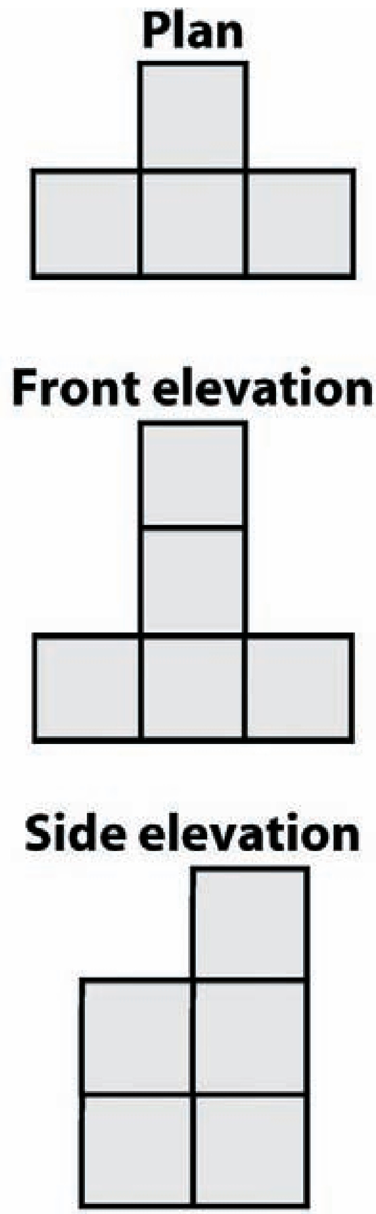


Isometric card L

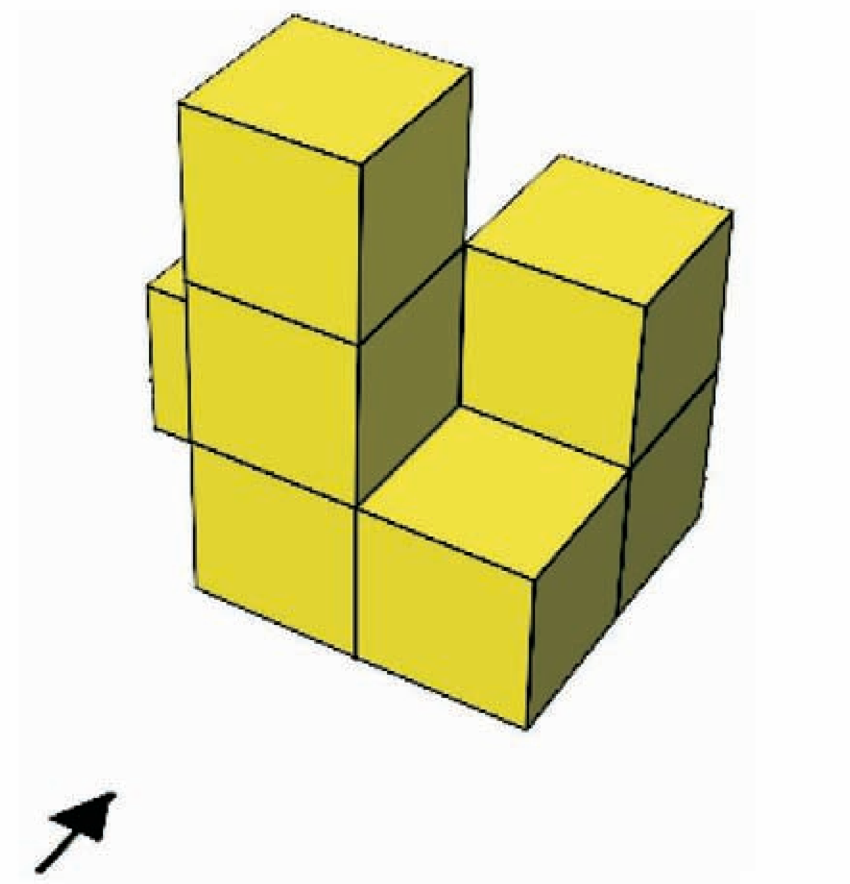


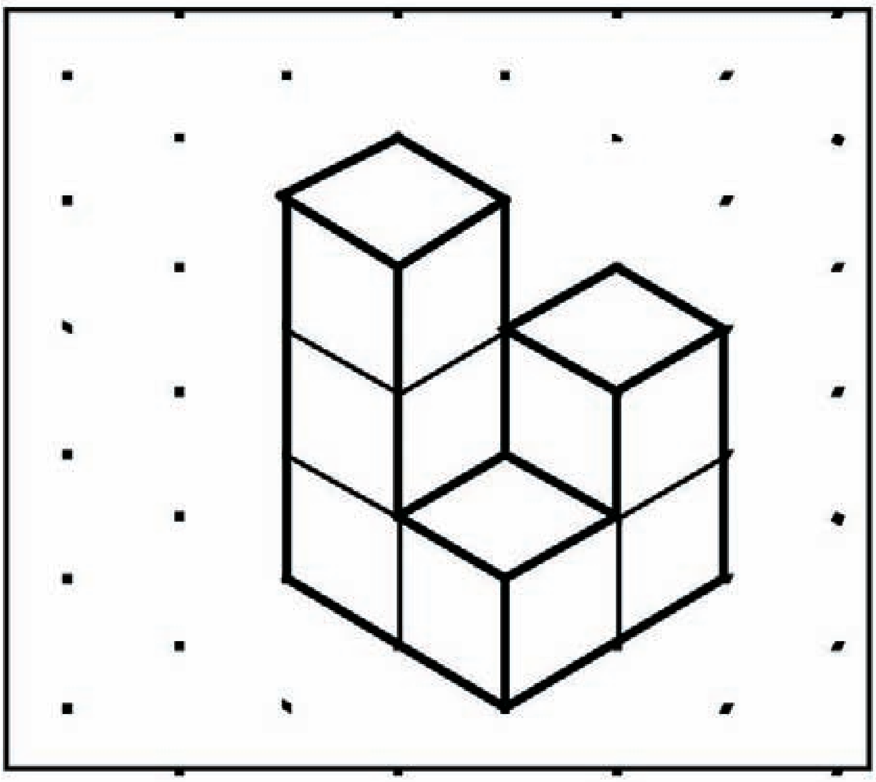
SS6 OHT 1 – *Plans and elevations*





SS6 OHT 2 – *Isometric drawing*





SS7 • Transforming shapes

Mathematical goals

To enable learners to:

- recognise and visualise transformations of 2D shapes;
- translate, rotate, reflect and combine these transformations.

Starting points

Learners unfamiliar with the terms ‘rotation’, ‘reflection’ and ‘translation’ will need some introduction to these. The session also assumes some acquaintance with equations of lines of reflection, e.g. $y = 4$, $x = 4$, $y = x$, $y = -x$. It is not essential, however, that all learners understand these ideas fully at the outset; during the session they will learn from each other through discussion.

Materials required

- OHT 1 – *Transformations*.

This should be cut in half so that the black L shape can be moved around over the grid to show different transformations.

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- a copy of OHT 1 – *Transformations*, printed on acetate;
- a pin (to help find centres of rotation);
- Card set A – *Transformation cards 1: pictures*;
- Card set B – *Transformation cards 2: words*;
- Card set C – *Additional cards: words and pictures* (2 pages).

Time needed

Approximately 1 hour.

Suggested approach

Beginning the session

Place the grid from OHT 1 – *Transformations* on an overhead projector. Place the L shape on top of the grid.

Ask learners where they think the image of the L shape will be after it has been translated/reflected/rotated in different ways. Discuss their responses. For each response, use the OHTs to demonstrate the correct position and how it can be found.

Working in groups

Ask learners to sit in pairs. Give out the copies of the OHT, pins, Card set A – *Transformation cards 1: pictures* and Card set B – *Transformation cards 2: words*.

The OHT should be cut horizontally along the centre of the page. Make sure that learners do not cut round the L shape as they will then find it difficult to find centres of rotation that lie outside the shape.

Explain that you would like learners to start by linking two *Picture* cards using a *Word* card. They should then try to link up more pairs and, ultimately, aim to end up with a connected network using as many of the cards as possible.

Encourage learners to explain carefully to each other why cards are linked and perhaps demonstrate this to you using the acetates on their table.

Learners who are likely to struggle should concentrate on linking pairs of pictures. If this proves very challenging they could be given a pair of pictures and asked to find the transformation that links them.

Learners who find the initial task easy could be given Card set C – *Additional cards: words and pictures*. They can use these cards to extend the networks they have already created.

When learners are happy with their results, ask them to compare their network with networks produced by other groups. It is easiest if cards are left on the tables and the learners move round. You can ask them to check other networks and to make notes of differences that emerge.

Reviewing and extending learning

Ask learners to do the following, using mini-whiteboards:

Show me the new coordinates of the point (1,4) after it is:

- reflected in the x axis; (1,–4)
- reflected in the y axis; (–1,4)
- rotated through 180° about (0,0); (–1,–4)
- reflected in the line $y = x$; (4,1)
- reflected in the line $y = -x$; (–4,–1)
- rotated 90° clockwise about (0,0); (4,–1)
- rotated 90° anticlockwise about (0,0). (–4,1)

You may like to repeat this with a general starting point (x,y).

You can extend the work to include combinations of transformations:

What is the single transformation that will produce the same result as:

- reflection in x axis, followed by reflection in y axis? (rotation 180° about (0,0))
- rotation 90° clockwise about (0,0) followed by reflection in y axis? (reflection in line $y = -x$)

Tracing paths round the cards laid on the table will help to generate and answer questions such as these.

What learners might do next

Learners may enjoy making up some cards of their own. They could use a different shape from the L and make up a small set of four cards showing the shape in different positions and four transformations that link them.

Further ideas

This session uses multiple representations of states and transformations. Similar activities in other mathematical contexts are included in this pack, e.g.

N7 Using percentages to increase quantities
(states are money values; transformations are percentage increases/decreases);

N8 Using directed numbers in context
(states are temperatures; transformations are rises and falls).

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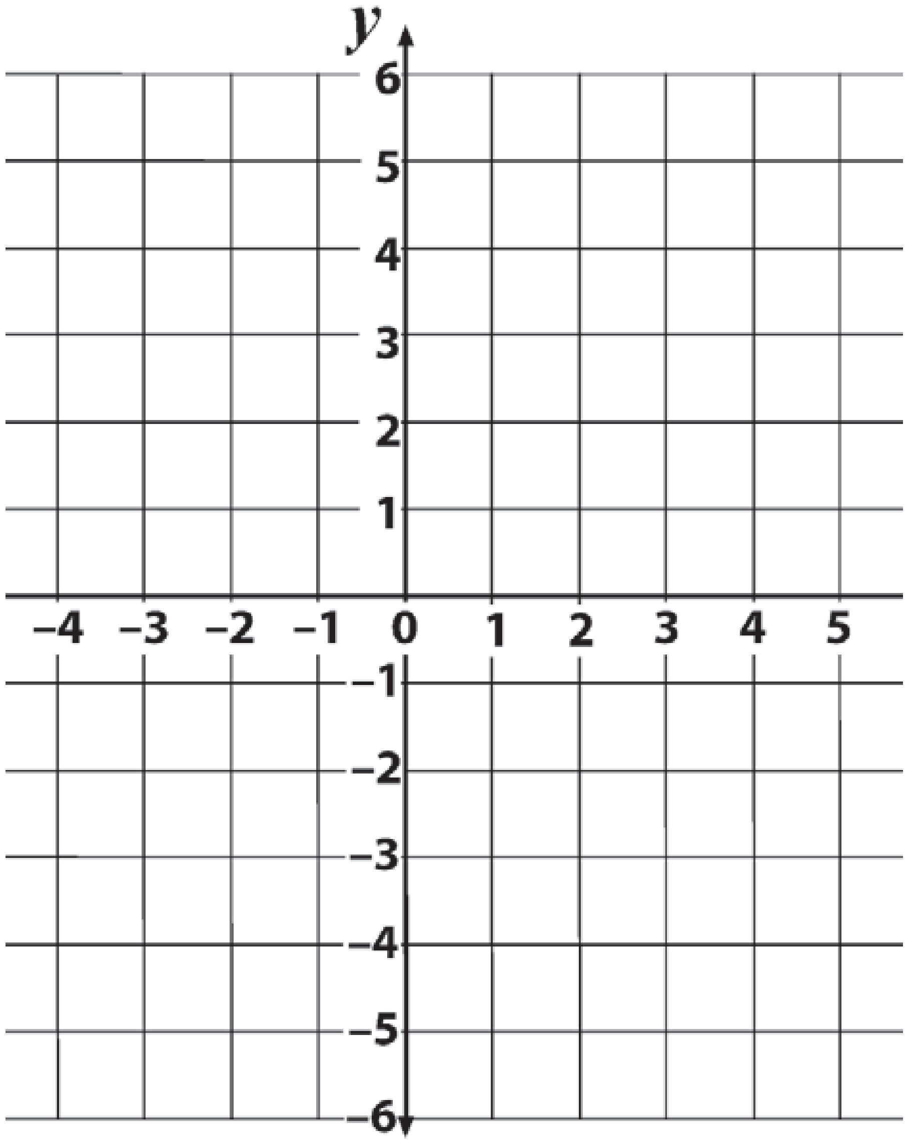
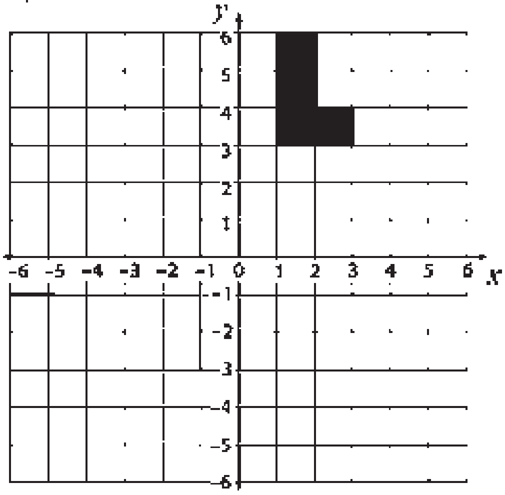
SS7 OHT 1 – *Transformations*

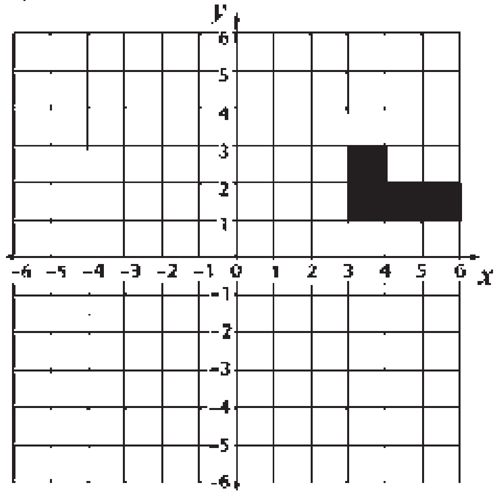


SS7 Card set A – *Transformation cards 1: Pictures*

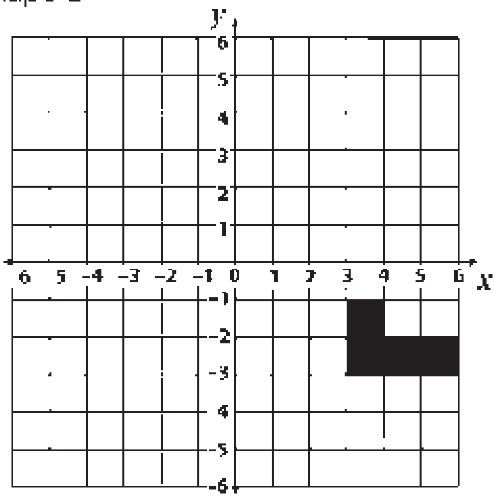
Shape A



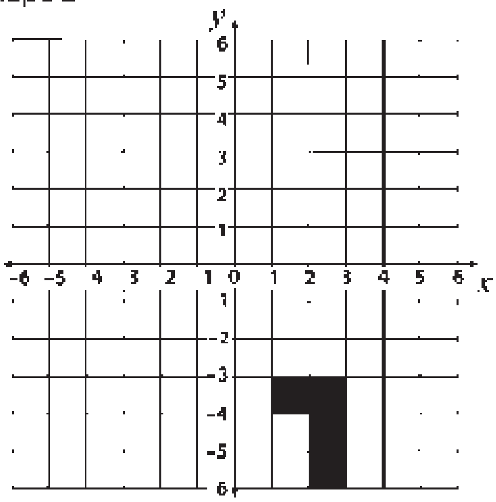
Shape B



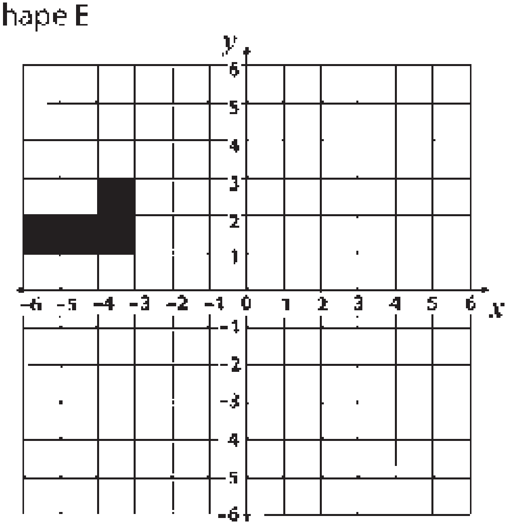
Shape C



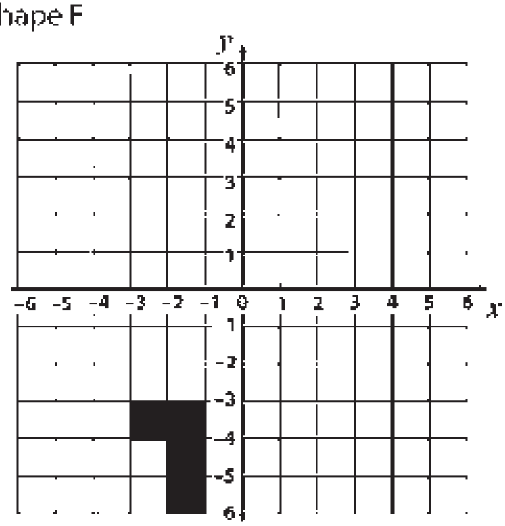
Shape D



Shape E



Shape F



SS7 Card set B – *Transformation cards 2: Words*

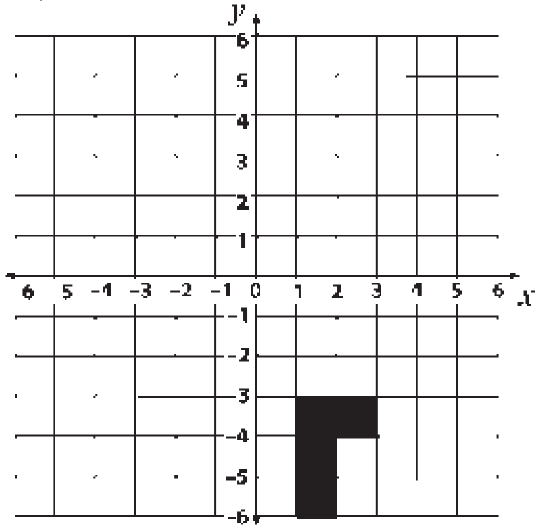
Reflect in the x axis	Reflect in the y axis
Reflect in the line $y = x$	Reflect in the line $y = -x$
Rotate 180° about (0,0)	Translate –4 units vertically
Rotate 90° clockwise about (0,0)	Rotate 90° anticlockwise about (2,2)

SS7 Card set C – *Additional cards: Words and pictures*

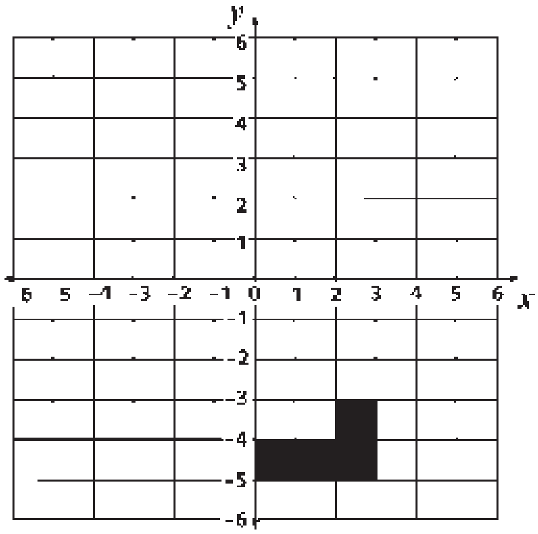
Rotate ....° clockwise about	Translate ...... horizontally and ...... vertically
Rotate ....° clockwise about	Reflect in the line ....
Translate ...... horizontally and ...... vertically	Reflect in the line ....

SS7 Card set C – *Additional cards: Words and pictures* (continued)

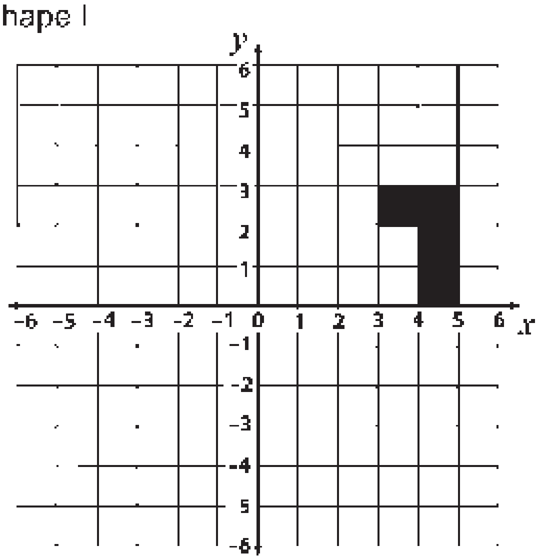
Shape G



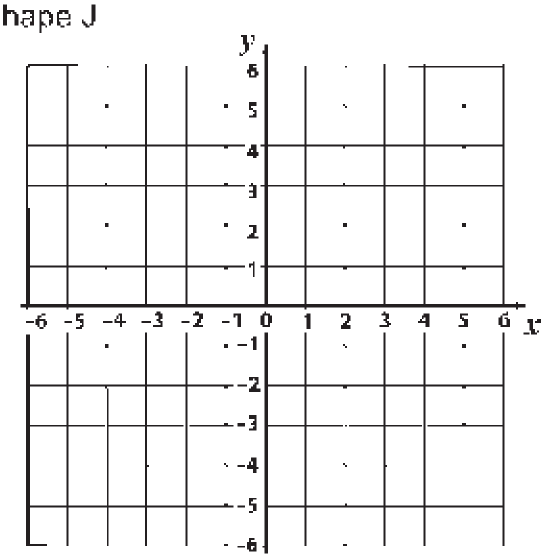
Shape H



Shape I



Shape J



SS8 • Developing an exam question: transformations

Mathematical goals

To help learners to:

- use past examination papers creatively;
- recognise and visualise transformations of 2D shapes;
- transform triangles and other 2D shapes by translation, rotation and reflection and combinations of these.

To develop learners' ability to:

- generalise and explore their own questions in this context.

Starting points

Most learners will have met the concepts of 'rotation', 'reflection' and 'translation' before, but may not have a deep understanding of them.

In this session, learners are provided with copies of a question similar to those found in GCSE examinations. They answer the question, then analyse the content and skills required to obtain the correct solution. They develop the task by asking further questions and by changing the task in various ways. They develop their own examination questions and attempt to answer the questions designed by other learners.

Materials required

- OHT 1 – *Transformations of a triangle*.

For each learner you will need:

- Sheet 2 – *Template for transformations*.

For each small group of learners you will need:

- Sheet 1 – *Transformations of a triangle*;
- a small supply of tracing paper or cut up blank acetate;
- pens that will write on acetate (if appropriate).

Time needed

Normally, this session will take about an hour.

Suggested approach

Beginning the session

Ask learners to work in pairs at the questions in Sheet 1 – *Transformations of a triangle*. Explain that these are similar to those found in GCSE examination papers. When everyone has had time to have a go, ask them to gather round for a whole group discussion about the questions.

Whole group discussion

(i) Answering the question

Begin by asking learners to define the terms used in the questions. As they do this, ask other learners to listen and suggest improvements. In particular, work on the meaning of the terms 'transformation', 'translation', 'reflection' and 'rotation'.

> A 'transformation' is a general term that describes a change in the position or shape of an object – in this case the shaded triangle. In this question we are considering three particular types of transformation.
>
> A 'translation' is the movement of a shape from one place to another in a straight line. Every point in the shape moves in the same direction and through the same distance.
>
> A 'reflection' is the movement of a shape so that it becomes a mirror image of itself. The reflection of each point in the original shape is perpendicular to the mirror line. If you fold your diagram along the mirror line, the original shape sits exactly on top of its reflection.
>
> A 'rotation' turns a shape about a fixed point called the 'centre of rotation'.

Collect suggestions for correct answers to each part of the question and write these on OHT 1 – *Transformations of a triangle*. If there are several different answers, ask learners to explain which they think is correct and why.

Although the question does not ask for explanations, it is important to get learners to describe their method for finding answers.

> How can you be sure that the reflection of the shaded triangle in the line $x = 3$ is triangle A?
> (If you fold the diagram along the line $x = 3$, the shaded triangle would sit exactly on top of triangle A.)
>
> How do you know that if the shaded triangle is rotated 90° clockwise about 0 it will sit exactly on top of triangle D?
> (I drew a line from the point (2, 4) to the right-angled corner of the shaded triangle. Then I rotated that line through 90° clockwise and reached the point (4, –2). This was the

right-angled corner of triangle D. I could repeat this for other points on the shaded triangle.)

Explanations may be helped by copying the shaded triangle onto a second, blank acetate and moving it over the first. In this way, translations, reflections and rotations can be shown dynamically.

Whole group discussion

(ii) Generating further questions

Ask learners to suggest alternative questions that could have been asked in this situation. Encourage learners to devise a range of questions, some easy and some more difficult. Also encourage them to devise questions that would test all the transformations. Make a list of these on the board.

For example:

Write down the letter of the triangle after the shaded triangle is:

- reflected in the line $y = 3$;
- rotated by 90° anticlockwise about 0;
- translated –3 units horizontally and –1 unit vertically.

Describe fully the single transformation that will take:

- triangle I to triangle J;
- triangle H to triangle J.

What triangles can be obtained from the shaded triangle by a single reflection, translation or rotation?

You can get from triangle E to H in two steps; a reflection in $y = -3$ (taking it to D) followed by a rotation about 0 through 180° (taking it to H). Find some different ways of getting from E to H using two transformations.

Working in groups

Ask learners to work in pairs to choose and answer questions from those listed on the board. Learners may like to compare their ideas using overhead transparencies or by writing on the board.

Whole group discussion

(iii) Developing the situation

Hand out Sheet 2 – *Template for transformations* to each learner. Ask learners to write a new transformation question by completing the sheet using diagrams and words. They should try to produce questions that are challenging, but which they think they can get right. They should write the answers on the back of the sheet.

For example, learners may suggest that the shaded triangle should be in a different position; they may also suggest that they would like to ask questions about particular kinds of transformations. They will then need to ensure that correct triangles appear on their diagram, together with some triangles in incorrect positions.

While doing this task, learners have to construct their own triangles to represent the outcomes of different transformations. This therefore extends the work that has gone before.

Working in groups

The new questions should be passed round the group to be answered by other learners. In cases of difficulties in answering questions, the question-writers should explain what they intended and act as a teacher helping other learners to answer the questions.

Alternatively, some of the new questions can be photocopied for further sessions or for homework.

Reviewing and extending learning

Finally, hold a whole group discussion on what has been learned, drawing out any common misconceptions. You should include a discussion of the level of difficulty of the new questions.

What learners might do next

Ask learners to choose another question from an exam paper and follow the process adopted in this session.

(i) Answer the question.

(ii) Generate new questions about the same situation (and answer them).

(iii) Change the situation and make up a new question.

Further ideas

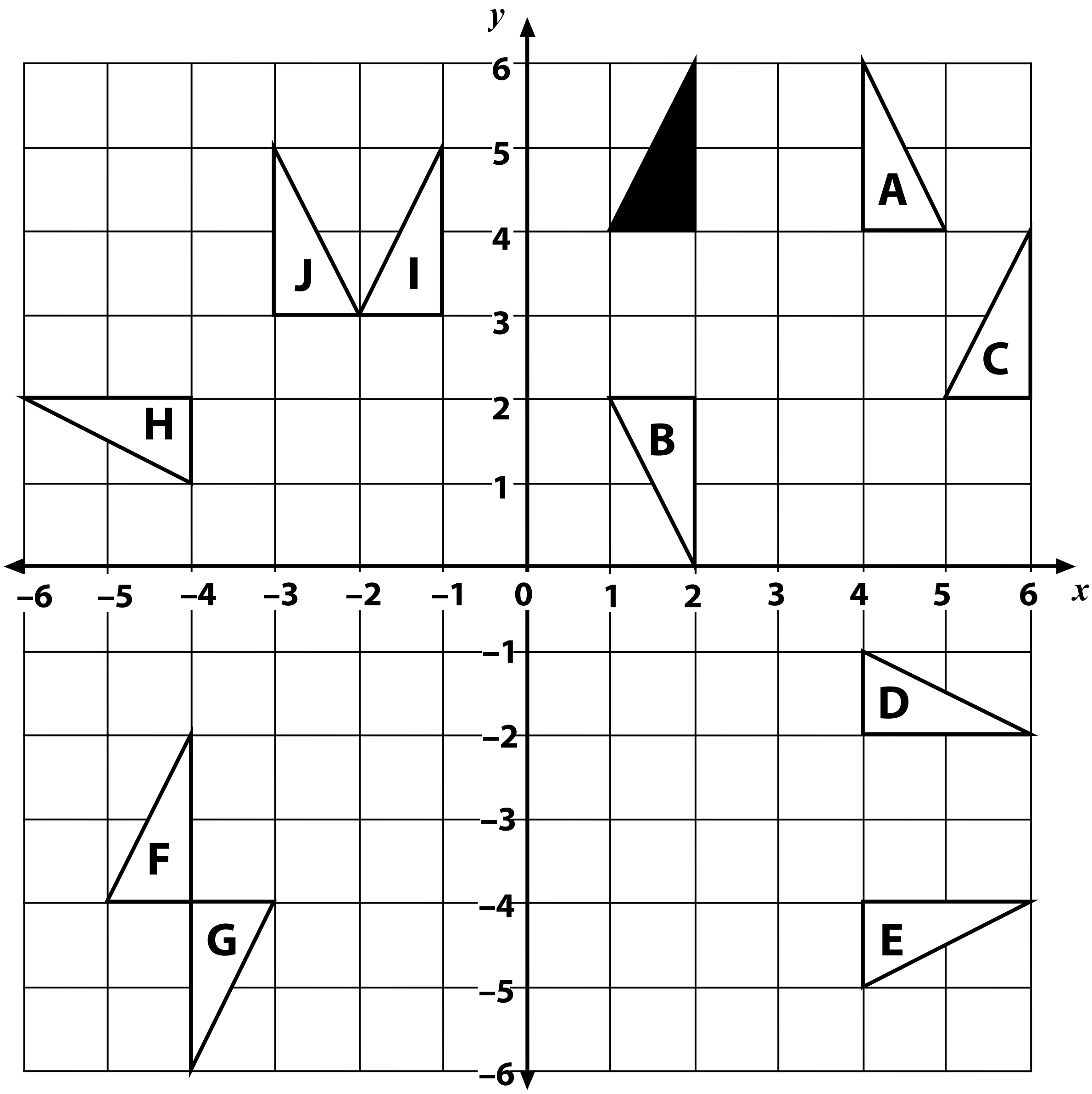
This method for developing exam questions may be used in any topic. Examples in this pack include:

N10 Developing an exam question: number;

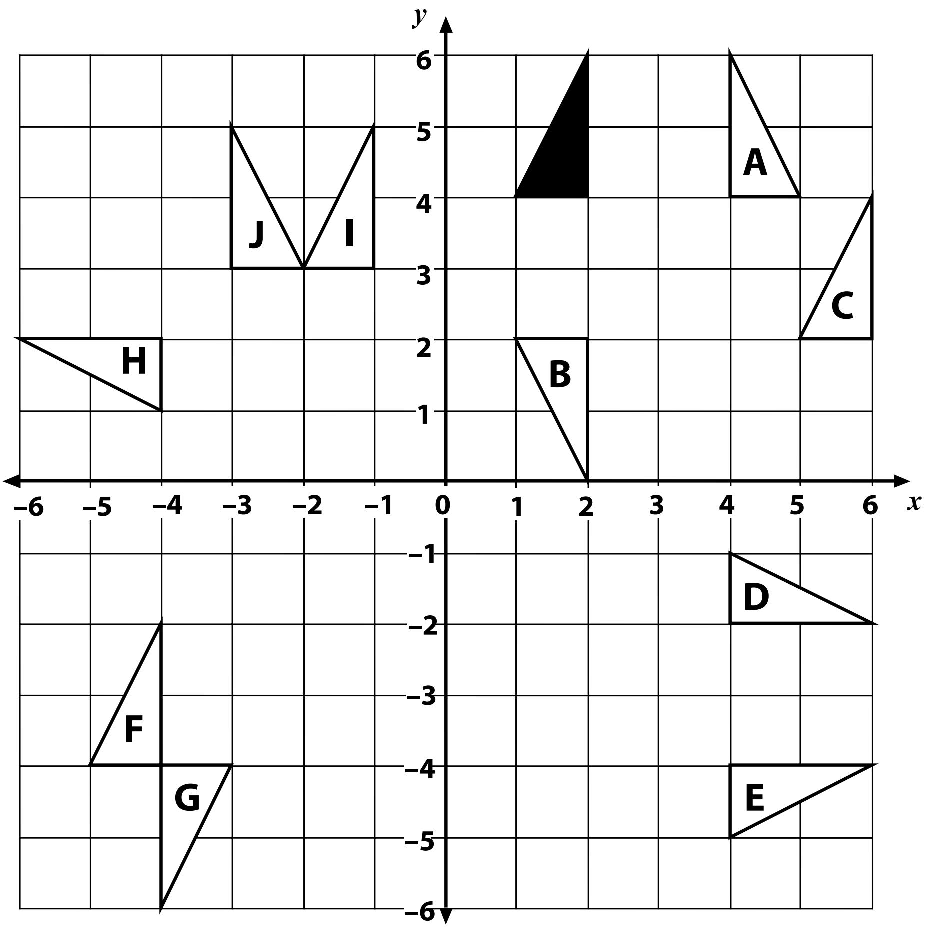
A8 Developing an exam question: generalising patterns;

S7 Developing an exam question: probability.

SS8 OHT 1 – *Transformations of a triangle*



SS8 Sheet 1 – *Transformations of a triangle*



1. Write down the letter of the triangle:

   (a) after the shaded triangle is reflected in the line $x = 3$;

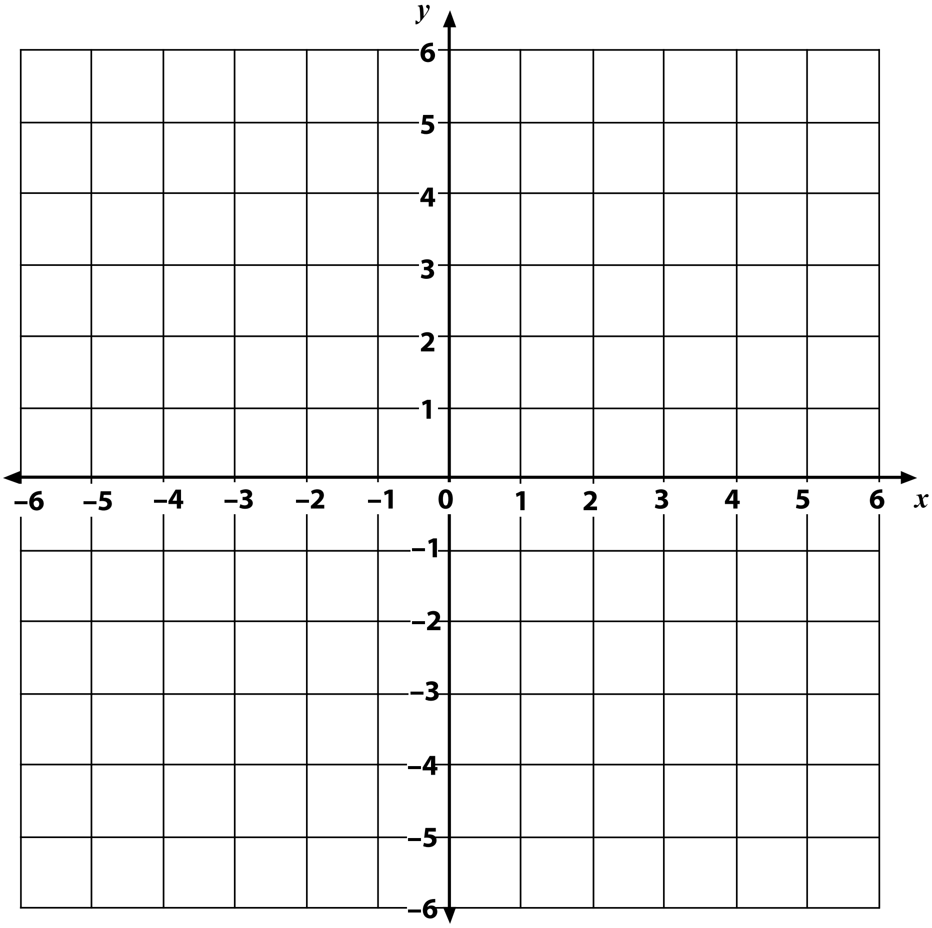
   (b) after the shaded triangle is translated by 4 squares to the right and 2 squares down;

   (c) after the shaded triangle is rotated 90° clockwise about 0.

2. Describe fully the single transformation that takes triangle F onto triangle G.

What other questions could have been asked?

SS8 Sheet 2 – *Template for transformations*



1. Write down the letter of the shaded triangle:
   (a) after it has been translated .... horizontally and .... vertically;
   (b) after it has been rotated through ....;
   (c) after it has been reflected in the line ....
2. Describe fully the single transformation that takes shape ...... onto shape ....
3. (Write your own harder question here)

....

....

....

....

....

....

S1 • Ordering probabilities

Mathematical goals

To help learners to:

- understand that probabilities are assigned values between 0 and 1.

To enable learners to:

- decide an appropriate value for the probability of a given event;
- use some of the vocabulary associated with probability such as 'certain', 'impossible', 'likely'.

To develop learners' ability to:

- order decimals between 0 and 1.

Starting points

Learners should have some experience of decimal numbers.

Materials required

Cards (approx. 10 cm by 8 cm) with events written on (for examples, see next page);

- long piece of string;
- pegs or sticky tape.

For each learner you will need:

- several blank cards;
- felt tip pen;
- long strip of paper.

Time needed

At least 30 minutes.

Suggested approach

Beginning the session

Fix the string across the room or along one wall. Onto one end, attach a card that says 'Certain' and onto the other end attach a card that says 'Impossible'. At intervals in between, attach cards that say 'Fairly likely', 'Very likely', 'Not very likely', 'Equally likely' and others, if wanted, using the language of probability. Ask a learner to arrange them into an appropriate order between the two extremes.

Whole group discussion

Pre-prepare some cards that refer to learners in the group and some that refer to events that are either topical or of particular interest to the learners, e.g.

- Kirsty will get her mobile phone out during this session;
- Hull City will get promotion;
- Mike will keep quiet for 10 minutes;
- Tomorrow's session will be cancelled;
- Aysha will arrive late to the next session.

Ensure that there is one card that refers to each learner, plus a few that refer to topical events. It is useful if one event card refers to something that is almost certain to happen and another to something that is almost impossible. Attach the topical event cards anywhere on the string between 'Certain' and 'Impossible'. Ask learners to discuss among themselves the order in which they should appear, depending on how likely they think that the event is. Learners should rearrange the cards accordingly.

When everyone is satisfied, give each learner the card that refers to him or her and ask him or her to place it on the probability line in a position that reflects how likely they think the event is.

Next, label the 'Impossible' card as 'Probability 0' and the 'Certain' card as 'Probability 1' and explain that probabilities are measured between 0 and 1. Ask for suggestions for numerical values (in decimals) for the topical card events. Write these values on blank cards and attach them above the event cards. The cards that relate to events that are 'almost certain' and 'almost impossible' should prompt discussion about how to get a very small number using decimals and how to get a number very close to 1.

Give every learner a blank card and ask them to write on it a decimal value of the probability of the event about themselves. Ask learners to attach the card above their event card.

Ask learners to look at the line and consider whether any events should be moved so that the probabilities are in the correct order. If learners are left to argue this out among themselves, without any teacher intervention, it can create some very interesting discussions about the size of decimals, e.g. which is the correct order for 0.8, 0.9 and 0.88; many learners have the misconception that '0.88 is bigger than 0.9'.

Reviewing and extending learning

Give a numerical value such as 0.75 and ask learners, working in pairs or groups, to suggest an event that could have that as the probability of its happening. Share the suggestions and invite comments as to whether they are appropriate for the given probability. Repeat this for other numbers.

Give each learner a long strip of card or paper and ask them to draw a probability line of their own, with events and associated probabilities of their own choice.

What learners might do next

Consider the probabilities of some other everyday events.

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S2 • Evaluating probability statements

Mathematical goals

To help learners to:

- discuss and clarify some common misconceptions about probability.

This involves discussing the concepts of:

- equally likely events;
- randomness;
- sample sizes.

Learners also learn to reason and explain.

Starting points

This session assumes that learners have encountered probability before. It aims to draw on their prior knowledge and develop it through discussion. It does not assume that they are already competent.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *True, false or unsure?*

Time needed

Between 30 minutes and 1 hour. The issues raised will not all be resolved in this time and will therefore need to be followed up in later sessions.

Suggested approach

Beginning the session

Using mini-whiteboards and questioning, remind learners of some of the basic concepts of probability. For example, ask learners to show you answers to the following:

Estimate the probability that:

- you will be hit by lightning this afternoon;
- you will get a tail with one toss of a coin;
- you will get a four with one roll of a die;
- you will sleep tonight.

Describe an event, different from those already mentioned, that has a probability of:

- zero;
- one;
- one half;
- more than one half, but less than one;
- less than one half, but greater than zero.

Working in groups

Give each pair of learners Card set A – *True, false or unsure?* Explain that these cards are intended to reveal some common misconceptions about probability.

Ask learners to take each card in turn and:

- decide whether it is a true statement or a false statement;
- write down reasons to support their decision;
- if they are unsure, explain how to find out whether it is true or not. For example, is there a simple experiment (simulation) or diagram that might help them decide?

As they do this, listen carefully to their reasoning and note down misconceptions that arise for later discussion with the whole group. When two pairs have reached agreement, ask them to join together and try to reach agreement as a group of four.

Whole group discussion

Ask each group of learners to choose one card they are certain is true and to explain to the rest of the group why they are certain. Repeat this with the statements that learners believe are false.

Finally, as a whole group, tackle the statements that learners are not so sure about.

Try to draw out the following points, preferably after learners have had the opportunity to do this in their own words.

- Statements B and H are true. For B it is enough to notice that there are two ways of obtaining a total of 3 (1,2 and 2,1), whereas there is only one way of obtaining a score of 2. For H, it is enough to notice that there are more learners than days of the week.

The remaining statements offer examples of common misconceptions.

- 'Special' events are less likely than 'more representative' events.

Statements A and C are indicative of this misconception. In both cases the outcomes are equally likely. Some learners remember trying to begin a game by rolling a six and it appeared to take a long time. The special status of the six has thus become associated with it being 'hard to get'. Others may think that they increase their chances in a lottery or raffle by spreading out their choices rather than by clustering them together. In fact this makes no difference.

- All outcomes are assumed to be equally likely.

Statements D and E are typical examples. The different outcomes are simply counted without considering that some are much more likely than others. For D, there are in fact four equally likely outcomes: HH, HT, TH, TT. Clearly, the probabilities for E will change whether the opposing team is Arsenal or Notts County.

- Later random events 'compensate' for earlier ones.

This is also known as the gambler's fallacy. Statements G and I are indicative of this. Statement G, for example, implies that the coin has some sort of 'memory' and later tosses will compensate for earlier ones. People often use the phrase 'the law of averages' in this way.

- Sample size is irrelevant.

Statement J provides an example of this subtle misconception. The argument typically runs that, if the probability of one head in two coin tosses is $\frac{1}{2}$, then the probability of n heads in $2n$ coin tosses is also $\frac{1}{2}$. In fact the probability of three out of six coin tosses being heads is $\frac{20}{64}$ or just under $\frac{1}{3}$. This may be calculated from Pascal's triangle.

- Probabilities give the proportion of outcomes that will occur.

Statement F would be correct if we replaced the word 'certain' with the words 'most likely'. Probabilities do not say for certain what will happen, they only give an indication of the likelihood of something happening. The only time we can be certain of something is when the probability is 0 or 1.

Learners who struggle with these ideas may like to do some simple practical probability experiments using coins and dice.

Reviewing and extending learning

Ask learners to suggest further examples that illustrate the misconceptions shown above.

What learners might do next

Session **S3 Playing probability computer games** may be used to follow up and deepen the ideas. This will make links between theoretical probabilities and experimental outcomes.

Further ideas

The idea of evaluating statements through discussion may be used at any level and in any topic where misconceptions are prevalent. Examples in this pack include:

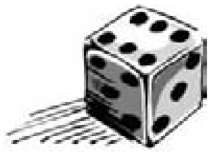
N2 Evaluating statements about number operations;

SS4 Evaluating statements about length and area.

S2 Card set A – *True, false or unsure?*

A

When you roll a fair six-sided die, it is harder to roll a six than a four.



B

Scoring a total of three with two dice is twice as likely as scoring a total of two.

C

In a lottery, the six numbers
3, 12, 26, 37, 44, 45
are more likely to come up than the six numbers 1, 2, 3, 4, 5, 6.

D

When two coins are tossed there are three possible outcomes: two heads, one head or no heads. The probability of two heads is therefore $\frac{1}{3}$.

E

There are three outcomes in a football match: win, lose or draw. The probability of winning is therefore $\frac{1}{3}$.

F

In a 'true or false?' quiz with ten questions, you are certain to get five right if you just guess.

G

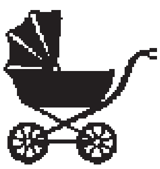
If you toss a fair coin five times and get five heads in a row, the next time you toss the coin it is more likely to show a tail than a head.

H

In a group of ten learners, the probability of two learners being born on the same day of the week is 1.

I

If a family has already got four boys, then the next baby is more likely to be a girl than a boy.



J

The probability of getting exactly three heads in six coin tosses is $\frac{1}{2}$.

S3 • Using probability computer games

Mathematical goals

To enable learners to:

- confront and overcome common misconceptions about probability;
- count equally likely outcomes, using diagrams;
- discuss relationships between theoretical probabilities, observed outcomes and sample sizes;
- calculate probabilities of dependent and independent events.

These goals may be adapted for learners aiming at lower levels. For example, you may decide to focus only on the first three goals.

Starting points

Learners will not need any prior knowledge about probability in order to use and analyse the games. This knowledge will be developed through the activity and the discussion that follows it.

It would however be helpful to use session **S2 Evaluating probability statements** before this one. This session will allow you to revisit some of the misconceptions that are described in that session.

Materials required

For each small group of learners you will need:

- Access to a computer loaded with the programs *Coin races* and *Dice races*;
- Sheet 1 – *Coin races*;
- Sheet 2 – *Recording sheet for coin races*;
- Sheet 3 – *Dice races*;
- Sheet 4 – *Recording sheet for dice races*.

Learners aiming at lower levels may find it more helpful to use just the first one or two games in each set.

Time needed

This is flexible, depending on the number of games used. As a rough guide, allow 1 hour for *Coin races* and 1 hour for *Dice races*. It is not essential that learners try to complete all the games. It is better to tackle two or three in depth than to cover them all superficially.

Suggested approach

Beginning the session

If you have a data projector or interactive whiteboard, it is very helpful to work through one race with the whole group. Before doing so, however, ask learners to make predictions about who will win and ask them to explain their reasoning.

As you work through a race, show learners how to fill in the recording sheet for that race. In particular, show them how to record the positions of the horses as the winner crosses the finishing line. Rather than simply writing the finishing order (1, 2, 3 etc.) it is more interesting to write the positions (the number of crosses in each row shows this) as the winner finishes the course.

Working in groups

Ask learners to work in pairs at each computer, making sure they have enough room to write down their results and their reasoning.

> Learners aiming at lower levels may find it helpful to work through some games using real dice and coins before they try to use the simulations.

Ask learners to work on one of the two situations, using the computer programs provided: *Coin races* or *Dice races*. The procedure for each set of situations is similar (see Sheets 1–4).

Learners are asked to:

1. Predict the outcome of each race before starting it. You may need to re-emphasise this as learners often find it hard to stop and think when they are working at a computer.
2. Run the race on the computer and record the outcome in a table. They should record the position of each horse when the winner reaches the finish. This should be repeated at least three times for each situation.
3. Reflect on what happened and try to explain this. For example:
   - Does the outcome vary very much from race to race?
   - Why does a particular horse seem to win more often than the others?
   - Why does a particular horse come last more often than the others?
   - Could the winning distance have been predicted?

Whole group discussion

After a set of races, collect together the results from the whole group and hold a discussion about the results. In the notes that follow, we have outlined one direction in which the discussion might go.

(i) Coin races

For the two-coin race, learners will have found that horse 1 wins most often, though the results vary for each race. If the group results are aggregated and average positions are found as the winner finishes, learners will find that, when horse 1 wins, the others have reached approximately the half-way stage. The race seems unfair. Ask learners why this should be so.

At this point, someone may mention the equally likely outcomes HH, HT, TH and TT and that this shows why horse 1 is twice as likely to move as the other horses. Explain that these outcomes may be shown using a list, a two-way sample space table or a tree diagram:

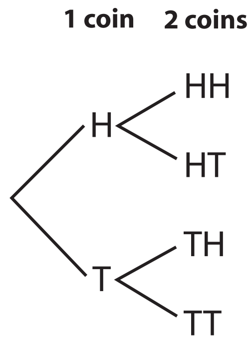
HH

HT

TH

TT

		Second coin	
		H	**T**
First coin	**H**	HH	HT
	T	TH	TT



Learners aiming at lower levels may find discussion of these representations helpful even if they do not go on to calculate probabilities.

Explain how this relates to the probabilities of each horse moving. Note that this is not the same as the probability of each horse winning. The longer the course, the more chance horse 1 has of winning. This relates to the 'sample size is irrelevant' misconception that is discussed in session **S2 Evaluating probability statements**.

Ask learners to draw tree diagrams for the three-horse situation in a similar way. They should then count the number of ways of getting 0, 1, 2 . . . heads for each number of coins and thus explain the likelihood of each horse winning.

They may like to list their observations and observe the patterns that emerge:

		Number of ways of getting						Total outcomes
		0 Heads	1 Heads	2 Heads	3 Heads	4 Heads	5 Heads	
Number of coins	1	1	1					2
	2	1	2	1				4
	3	1	3	3	1			8
	4	1	4	6	4	1		16
	5	1	5	10	10	5	1	32

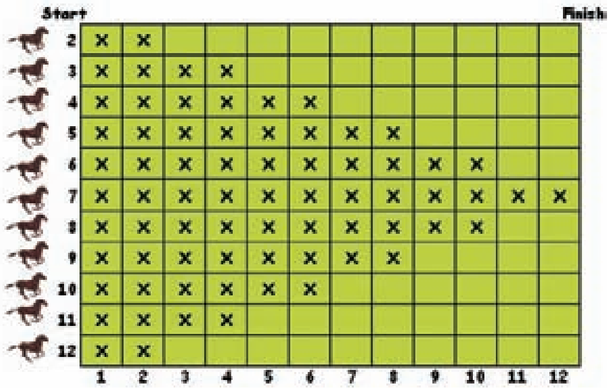
The probability of each horse winning can be calculated directly from this table. Thus with five coins, the probability that horse number 2 ('2 heads') wins is $\frac{10}{32} = \frac{5}{16}$. Learners may be able to predict further rows of this table from those shown here.

Relate this to the outcomes observed. Was it true that, for three coins, horses 2 and 3 won most often and were approximately three times as far down the course as the other two horses? What if the results of the whole group were aggregated?

(ii) Dice races

In the Sums race, learners should notice that the horses in the centre of the field tend to win more often than those at the outside. Can they explain this? Refer again to the sample space diagrams that display equally likely outcomes:

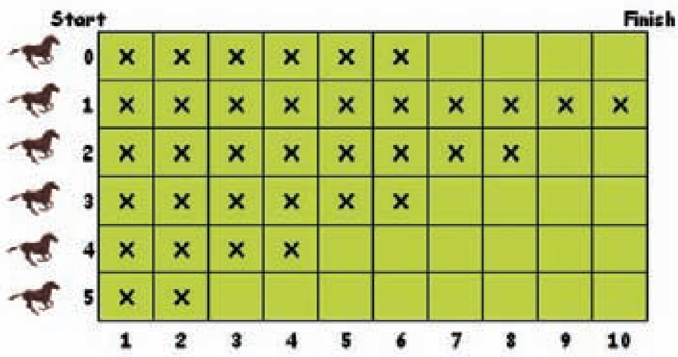
Sums race		First die					
		1	2	3	4	5	6
Second die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12



The diagram on the left shows, for example, that we would expect horse 7 to move six times more often than horse 2. Show learners how to calculate, from this diagram, the probabilities that particular horses will move. Can learners describe how this table gives a prediction of the positions of horses when horse 7 finishes? (This is shown in the diagram on the right.)

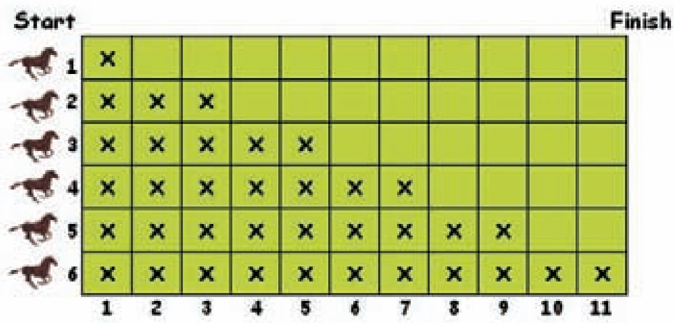
Now learners can be asked to analyse the remaining two situations in a similar manner, using sample space diagrams. They should find that the results from the Differences race give an asymmetric distribution, with horse 1 the most likely to win.

Differences race		First die					
		1	2	3	4	5	6
Second die	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0



They should find that the results from the Max dice race also give an asymmetric distribution, with horse 6 the most likely to win.

Max race		First die					
		1	2	3	4	5	6
Second die	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6



The Multiples dice race is slightly more complicated as a particular throw of the dice may result in two or more horses moving at one time. Thus a throw of $3 \times 4 = 12$ will result in horses 2, 3, 4 and 6 all moving. Ask learners to tell you which of horses 6 and 2 will be the faster and why.

The easiest place to begin is by simply writing out a list of possible products. Learners can then count the multiples systematically.

Multiples race		First die					
		1	2	3	4	5	6
Second die	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

There are:

27 multiples of 2 and so prob (mult of 2) = $\frac{27}{36}$

20 multiples of 3 and so prob (mult of 3) = $\frac{20}{36}$

15 multiples of 4 and so prob (mult of 4) = $\frac{15}{36}$

11 multiples of 5 and so prob (mult of 5) = $\frac{11}{36}$

15 multiples of 6 and so prob (mult of 6) = $\frac{15}{36}$

Note that the probabilities do not add up to 1.

This enables us to predict the outcome: horse 2 should win, followed by horse 3, then 4 and 6 (together), with 5 bringing up the rear.

In each of the above situations, learners may question why their own results do not correspond to the theory. This is a good time to consider the issue of sample size. Small samples may not correspond to these results but, aggregated over the whole group, the results should correspond more closely.

Reviewing and extending learning

Ask learners to analyse a game that they haven't yet used. For example, you could ask them to analyse a game where they have one coin and one die. If they throw a head, they double the number thrown on the die, otherwise the number stands. Can learners work out the possible outcomes and their probabilities?

What learners might do next

Learners may enjoy developing their own GCSE probability question. A suitable session, **S7 Developing an exam question: probability**, is provided in this pack.

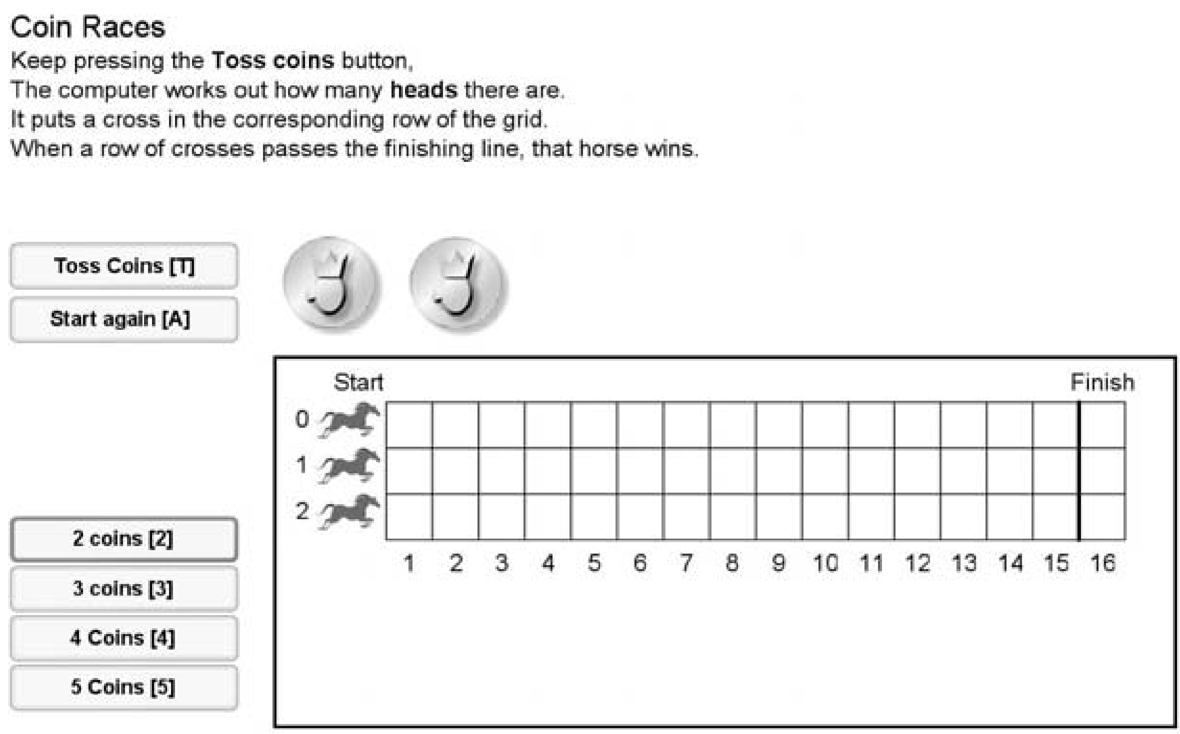
Further ideas

This activity uses multiple representations to deepen understanding of probability. Learners may find it helpful to devise sample space and tree diagrams for their own dice, coin and spinner situations.

S3 Sheet 1 – *Coin races*

You will be using the computer for these tasks.
Start with the two-coin race and work through the other races in order.
You should write your results in the tables on your recording sheet.

1. Before you start each race, write down the positions you predict horses will be in as the winner crosses the finishing line.
   Why do you think this?
   Write down your reason next to the table in the space provided.



2. Run each race several times.
   Each time, record the positions of the horses as the winner crosses the finishing line.
   (Count how many crosses there are in each row).
3. Describe and explain any patterns you find in your data.

Repeat steps 1–3 for the three-, four- and five-coin races.

S3 Sheet 2 – *Recording sheet for coin races*

2 coin tosses

	Horse number (number of heads thrown)		
	0	1	2
Your prediction			
First race actual finishing positions			
Second race actual finishing positions			
Third race actual finishing positions			

3 coin tosses

	Horse number			
	0	1	2	3
Your prediction				
First race actual finishing positions				
Second race actual finishing positions				
Third race actual finishing positions				

4 coin tosses

	Horse number				
	0	1	2	3	4
Your prediction					
First race actual finishing positions					
Second race actual finishing positions					
Third race actual finishing positions					

5 coin tosses

	Horse number					
	0	1	2	3	4	5
Your prediction						
First race actual finishing positions						
Second race actual finishing positions						
Third race actual finishing positions						

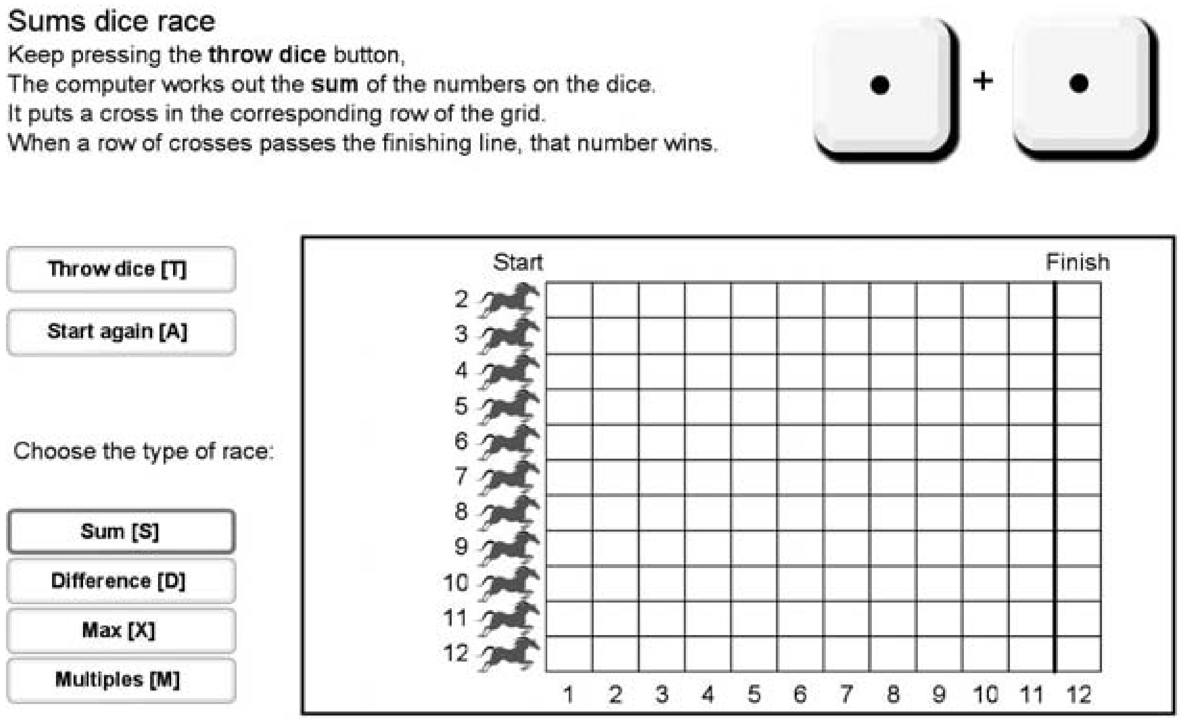
Space for your reasoning

In order to draw sensible conclusions, you will need to put these results together with those from the rest of the group.

S3 Sheet 3 – *Dice races*

You will be using the computer for these tasks.
Start with the Sums race and then work through the other races in order.
You should write your results in the tables on your recording sheet.

1. Before you start each race, write down the positions you predict the horses will be in as the winner crosses the finishing line.
   Use the row of the table that is labelled 'Your prediction'.
   Why do you think this?
   Write down your reason next to the table in the space provided.



2. Run each race several times.
   Each time, record the positions of the horses as the winner crosses the line. (Count how many crosses there are in each row.)
3. Try to explain any patterns you find in your data.

Repeat steps 1, 2 and 3 for the Difference, Max and Multiples races.

S3 Sheet 4 – *Recording sheet for dice races*

Addition race

	Horse number										
	2	3	4	5	6	7	8	9	10	11	12
Your prediction											
First race actual finishing positions											
Second race actual finishing positions											
Third race actual finishing positions											

Difference race

	Horse number					
	0	1	2	3	4	5
Your prediction						
First race actual finishing positions						
Second race actual finishing positions						
Third race actual finishing positions						

Max race

	Horse number					
	1	2	3	4	5	6
Your prediction						
First race actual finishing positions						
Second race actual finishing positions						
Third race actual finishing positions						

Multiples race

	Horse number				
	2	3	4	5	6
Your prediction					
First race actual finishing positions					
Second race actual finishing positions					
Third race actual finishing positions					

Space for your reasoning

In order to draw sensible conclusions you will need to put these results together with those from the rest of the group.

S4 • Understanding mean, median, mode and range

Mathematical goals

To help learners to:

- understand the terms: mean, median, mode, range;
- explore the relationships between these measures and their relationship to the shape of a distribution.

Starting points

Most learners will have met the terms mean, median, mode and range but they may not have a clear understanding of their meaning and the relationships between them. One purpose of this session is to expose and discuss any misconceptions.

Materials required

An overhead or data projector, or an interactive whiteboard, can be very helpful during the early part of the session.

- OHT 1 – Recording data;

or

- computer program *Statistics 1* provided on the DVD-ROM/CD.

For each small group of learners you will need:

- Card set A – *Bar charts*;
- Card set B – *Statistics*;
- large sheet of paper for making a poster;
- glue stick;
- access to a computer loaded with the computer program *Statistics 1* (optional).

You may feel it appropriate for learners aiming at lower levels to be given just the first six cards in each set.

Time needed

Approximately 1–2 hours.

Make sure you leave plenty of time for the closing discussion on methods.

Suggested approach

Beginning the session

Ask the group to give you 11 numbers, or 'scores', between 1 and 6. As learners call them out, write them in the *Raw scores* box in OHT 1. Tell learners that these numbers could, for example, represent numbers that were thrown on rolls of a die or from some other experiment.

Ask learners to show you how a frequency table and then a bar chart can be drawn from this raw data. This should be fairly straightforward. Learners may like to come out and fill in some of the OHT for you.

Ask the group to explain the meaning of the terms 'mean', 'median', 'mode' and 'range'. As they do so, ask learners to come out and demonstrate the following calculations using the raw scores.

> The 'mean' is calculated by summing all the data and dividing by the number of terms.
>
> The 'median' is calculated by putting the scores in order of size and then finding the middle score.
>
> The 'mode' is found by looking for the most common score.
>
> The 'range' is found by calculating the difference between the greatest and least scores.

Hide the raw scores and the bar chart and ask learners how they might have calculated these statistics by looking only at the frequency table.

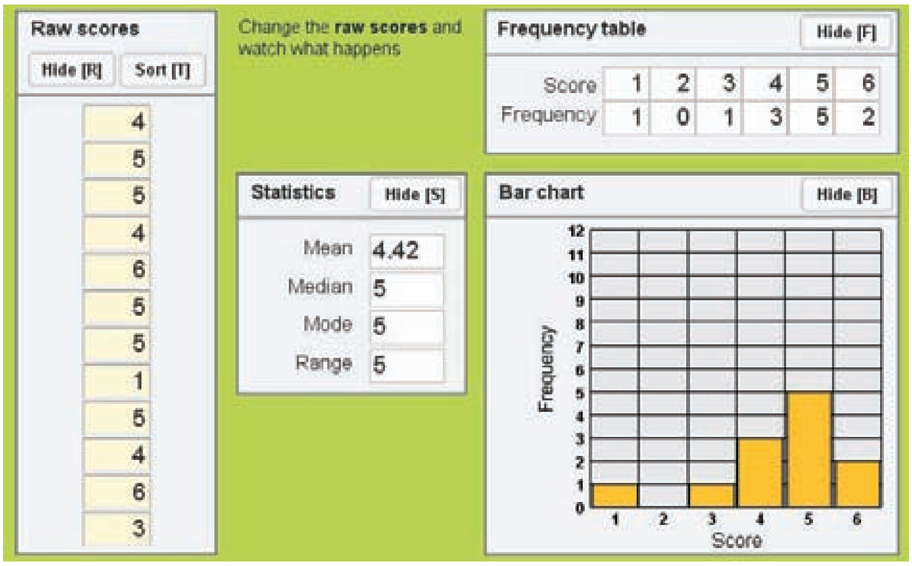
> When calculating the mean, what is wrong with just adding all the frequencies and dividing by 6?
>
> How can we find the median by looking only at the frequency table?

Finally, hide both the raw scores and the frequency table and ask the same questions, referring only to the bar chart.

> Without writing anything down, how can we calculate the median, the mode and the range from the bar chart?
>
> How can we calculate the mean?

Using an interactive whiteboard

A computer program is provided on the DVD-ROM to facilitate this discussion. To begin with, we suggest that you hide the frequency table, the bar chart and the statistics, and then reveal these in stages as the explanation proceeds.



Working in groups

Give Card set A – *Bar charts* and Card set B – *Statistics* to each pair of learners.

Ask learners to work together, trying to match pairs of cards. They will notice that some of the *Statistics* cards have gaps on them and one of the *Bar charts* cards is blank. Learners should try to work out what these blanks should be.

As they work on this task, encourage learners to take turns at explaining how they know that particular cards match. Allow learners a period of time to get to know the cards, and to think of a good approach to the problem.

Learners who struggle with the task may be helped by some strategic hints:

> Try sorting the two sets of cards into order first – from smallest to highest range.
> Now repeat this using the modal values . . .
> Now the medians . . .

If learners continue to struggle, encourage them to write out the raw scores from the bar chart and then sort and organise them in order to calculate the statistics.

During the discussion, the following questions may promote deeper thinking:

> Which two bar chart cards show a sample size of 12?
> (J and K).
>
> Can we tell how big the sample size is from the statistics?
> (No)
>
> How do we work out the median when there is no middle number?
> (This occurs on bar chart J, where the median is 3.5).

It may be worth stopping the session at some point to discuss these issues with the whole group.

Reviewing and extending learning

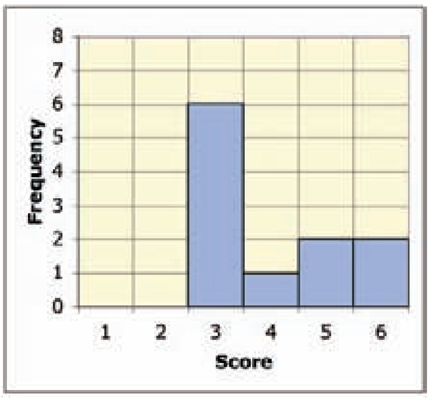
Learners may like to make posters of their finished work by pasting the cards onto a large sheet of paper.

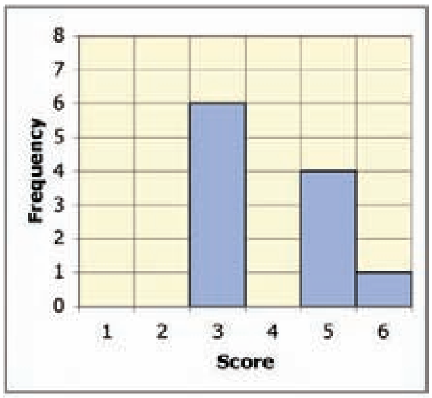
If you have the computer software available, learners could check their solutions by entering raw scores that correspond to a particular bar chart and then check to see that the mean, median and mode are correct.

In the whole group discussion, discuss some of the strategies that learners used to sort the cards.

> The mode was the easiest, so I put all the cards with the same mode together. I then looked at the range. I then looked at the median and finally the mean.

Finally, discuss how the missing bar chart may be constructed from the statistics on *Statistics* card Stats G. Learners may be surprised that there is more than one correct answer. For example, both these bar charts have mean = 4, median = 3, mode = 3, range = 3. In addition, we have no idea what the sample size is.





One approach might be:

> I first saw that the mode was 3, so there must be a tall bar by a score of 3. The fact that the median is also 3 means that the middle score is at 3. This made me think that there might be 6 scores at 3. If there were many lower than 3, I would have trouble getting a mean greater than 3. As the range is 3, then I thought that the scores would go from 3 to 6.

What learners might do next

When learners have reached a clear understanding of mean, median, mode and range, they may be ready to move on to another statistics session, **S5 Interpreting bar charts, pie charts, box and whisker plots**.

Further ideas

This activity uses multiple representations to deepen understanding of statistical measures. This type of activity can be used in any topic where a range of representations is used. Examples in this pack include:

N5 Understanding the laws of arithmetic;

A1 Interpreting algebraic expressions;

SS6 Representing 3D shapes.

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S4 OHT 1 – *Recording data*

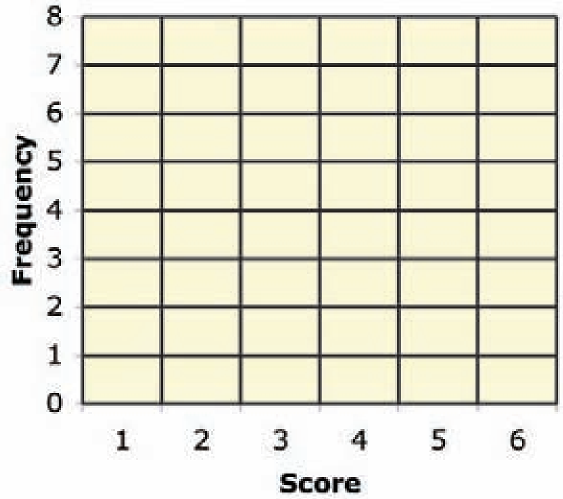
Raw scores

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Frequency table

Score	1	2	3	4	5	6
Frequency						

Bar chart

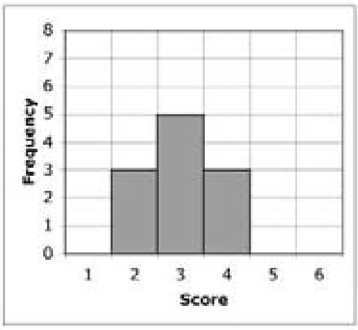


Statistics

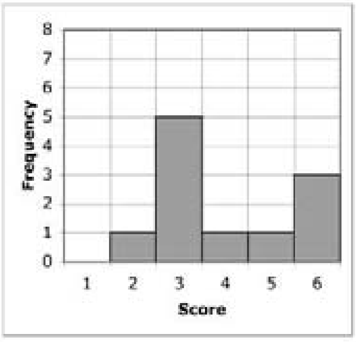
Mean	
Median	
Mode	
Range	

S4 Card set A – *Bar charts*

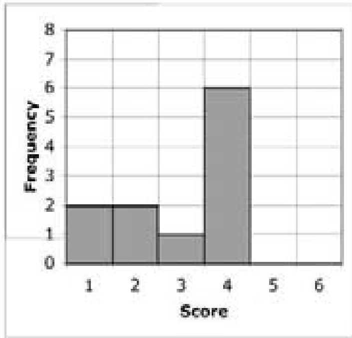
Bar chart A



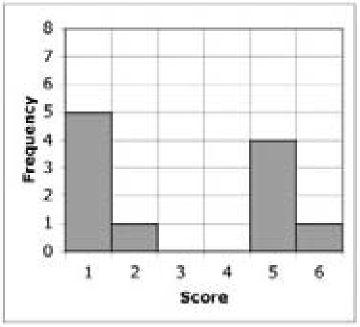
Bar chart B



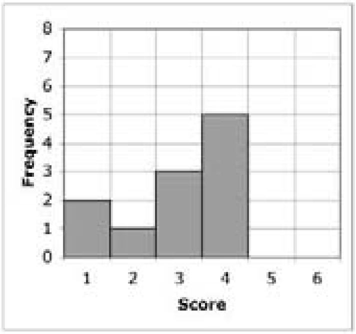
Bar chart C



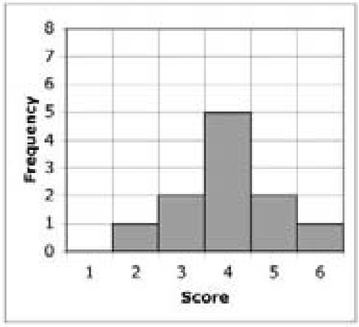
Bar chart D



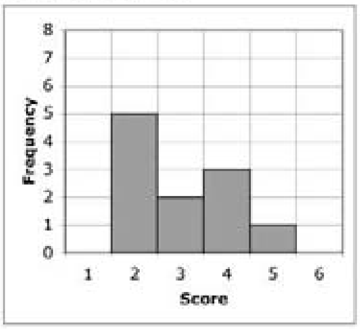
Bar chart E



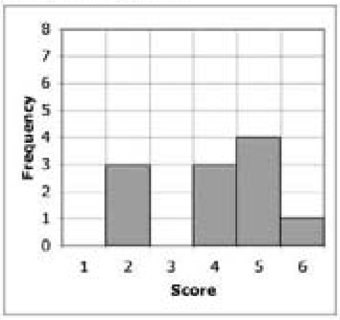
Bar chart F



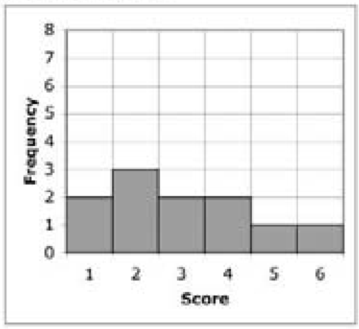
Bar chart G



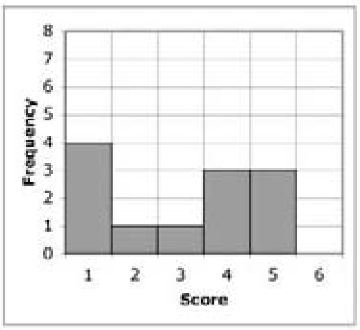
Bar chart H



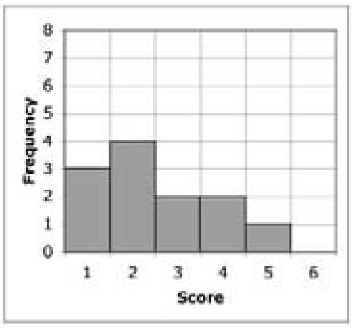
Bar chart I



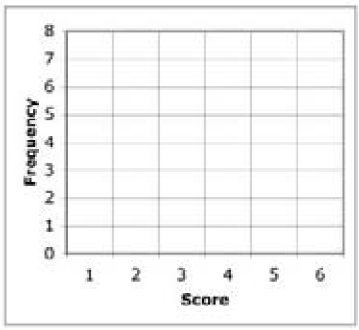
Bar chart J



Bar chart K



Bar chart L



S4 Card set B – *Statistics*

Stats A

Mean	3
Median	4
Mode	4
Range	3

Stats B

Mean	3
Median	3
Mode	3
Range	

Stats C

Mean	3
Median	2
Mode	
Range	5

Stats D

Mean	4
Median	4
Mode	4
Range	4

Stats E

Mean	3
Median	3
Mode	4
Range	3

Stats F

Mean	
Median	3
Mode	3
Range	4

Stats G

Mean	4
Median	3
Mode	3
Range	3

Stats H

Mean	
Median	2
Mode	2
Range	4

Stats I

Mean	3
Median	
Mode	2
Range	3

Stats J

Mean	3
Median	
Mode	1
Range	4

Stats K

Mean	3
Median	3
Mode	
Range	5

Stats L

Mean	4
Median	4
Mode	5
Range	

S5 • Interpreting bar charts, pie charts, box and whisker plots

Mathematical goals

To help learners to:

- understand and interpret bar charts, pie charts, and box and whisker plots.

Starting points

This session is in two linked parts.

- Matching pie charts to bar charts.
- Matching box and whisker plots to bar charts.

Each part of the session starts with a whole group discussion to compare the newly-introduced type of representation, looking at its advantages, disadvantages and practical applications. Learners then work in pairs.

No prior knowledge is assumed, though it is helpful if learners have encountered some of these ideas before.

If computers are available, solutions may be checked using the computer program *Statistics 2* that is provided on the DVD-ROM/CD.

Materials required

An overhead projector or data projector is very helpful during the introduction.

For each small group of learners you will need:

- Card set A – *Bar charts*;
- Card set B – *Pie charts*;
- Card set C – *Box and whisker plots*;
- Card set D – *Making your own cards*;
- OHT 1 – *Statistical representations* (optional);
- The computer program *Statistics 2* (optional).

Time needed

Approximately 2 hours.

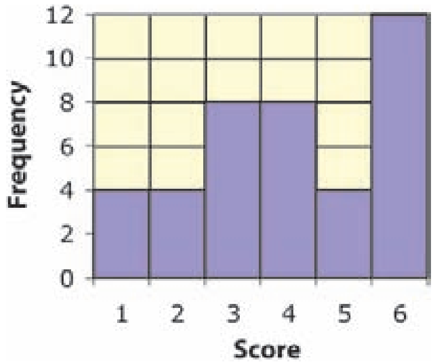
The session is in two linked parts. Each part will take up to 1 hour.

Suggested approach

Beginning the session

Introduce the session using OHT 1 – *Statistical representations*, or using the computer program *Statistics 2* and a data projector. Start by drawing on the board the bar chart shown here. If a data projector is used, display the bar chart using the program *Statistics 2* and hide everything except the bar chart.

This bar chart represents the scores that were obtained when a number of people entered a penalty-taking competition. Each person was allowed six penalty kicks.



How many people entered the competition?
How can you tell?

How can you calculate the mean, median and modal number of penalties scored?

What proportion of the people scored one penalty?
What is that as a percentage?
What proportion scored three penalties? Six penalties?

Can you think of another type of statistical diagram that can be used to show proportions?

You can use this to introduce the idea of a pie chart. Ask learners to sketch one if they can, then show them how this can be done. Focus attention on the pie chart through careful questioning. For example:

Does the pie chart tell you how many people entered the competition?
No? So what does it tell you?

How can you find the mode and median from the pie chart?

Can you estimate the percentage that scored six goals?

If only four people had scored six goals, what would the pie chart have looked like?

If I halve/double the heights of all the bars in the bar chart, what will happen to the pie chart?

Try to draw out from learners the relative advantages and different uses of bar charts and pie charts , e.g. bar charts help you to see the shape of the distribution and give you more data, including the numbers involved. Pie charts help you to see the proportions (or fractions) of the total in each category.

Working in groups (1)

Hand out Card set A – *Bar charts* and Card set B – *Pie charts* to each pair of learners. Ask learners to match the cards from each set.

As they work on this task, encourage learners to explain how they know that particular cards match. When learners are stuck, ask questions that might help them to develop a strategy.

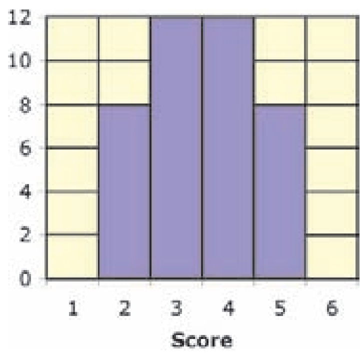
> Which bar charts have the smallest range?
> How is the range shown on the pie chart?
>
> What is the modal score on the bar chart?
> Which pie charts have the same mode?

If some learners complete the matching task quickly, give them copies of Card set D – *Making your own cards* and ask them to devise two matching card sets of their own.

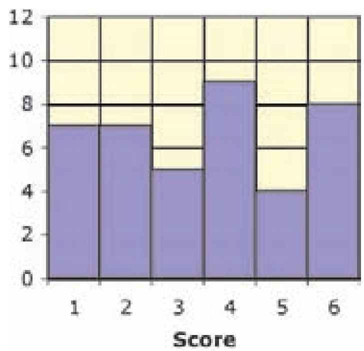
Whole group discussion

Draw the two bar charts shown here on the board.

Wine A

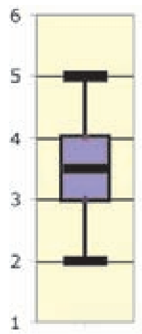


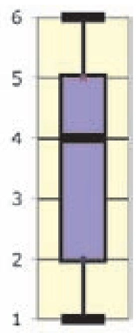
Wine B



> Forty people are asked to taste two types of wine. Each is asked to rate the wine on a scale from 1 to 6. 1 = awful, 6 = fantastic. The graphs show the results of the wine tasting. What can you say about the wines? If you had someone coming to dinner, which wine would you choose? Why?

Both wines have the same mean score, 3.5. People share a similar view about wine A, but they have a wide spread of views about wine B. There is a statistical diagram that is helpful when making comparisons of spread: the 'box and whisker' plot.





Explain the five data points that are used to construct the box and whisker plot:

- the least and greatest values (the whiskers);
- the median (the middle line);
- the quartiles (the ends of the boxes).

Explain that box and whisker plots can be drawn vertically or horizontally.

For wine A, the range is from 2 to 5, the median is 3.5 (20 scores are above and 20 are below this value) and the quartiles are at 3 and 4 (when the 40 scores are placed in order, the 10th score is 3 and the 30th is 4).

For wine B, the range is from 1 to 6, the median is 4, and the quartiles are at 2 and 5.

Working in groups (2)

Hand out Card set A – *Bar charts* and Card set C – *Box and whisker plots* to each pair of learners.

Ask learners to work together to match the cards from each set. They should try to do this without doing calculations.

As learners work on this task, encourage them to take turns at explaining how they know particular cards match. When learners are stuck, ask questions that might help them to look at the overall structure.

Can you sort the cards into those that have a large range and those that have a small range?

Can you sort the cards into those that have a large median and a small median?

Does the distribution look spread out (the 'box' is large), or is it concentrated in a few scores (the 'box' is small)?

Does the distribution look symmetrical, or is it skewed?

Reviewing and extending learning

Show OHT 1 – *Statistical representations* (or use the *Statistics 2* software provided) and ask learners questions to review the session.

For example:

- Show a frequency table and ask learners to predict what the bar chart, pie chart and/or the box and whisker plot will look like.
- Show just a pie chart and ask for a suitable bar chart.
- Show just a box and whisker plot and ask for a suitable bar chart.

. . . and so on.

In each case approximate answers, with reasons, will be sufficient.

What learners might do next

S6 Interpreting frequency graphs, cumulative frequency graphs, box and whisker plots is a good follow-up to this session.

Further ideas

This activity uses multiple representations to deepen understanding of statistical measures. This type of activity can be used in any topic where a range of representations is used. Examples in this pack include:

N5 Understanding the laws of arithmetic;

A1 Interpreting algebraic expressions;

SS6 Representing 3D shapes.

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S5 OHT 1 – *Statistical representations*

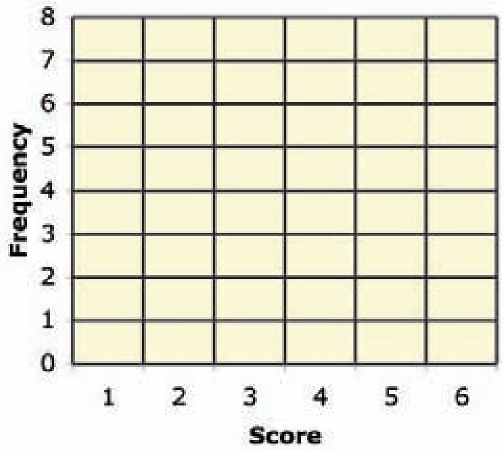
Frequency table

Score	1	2	3	4	5	6
Frequency						

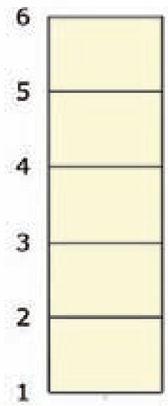
Statistics

Mean	
Median	
Mode	
Range	

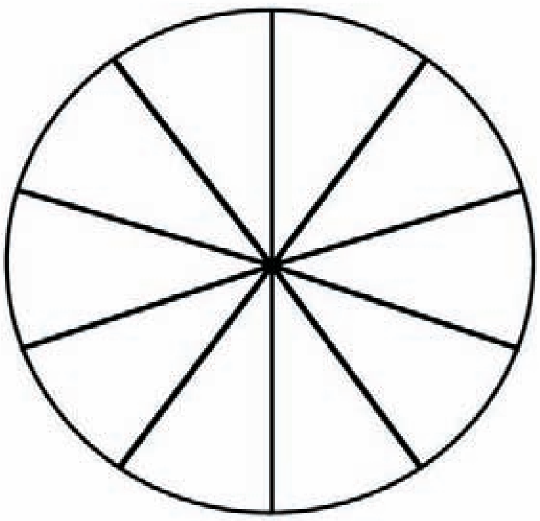
Bar chart



Box and whisker

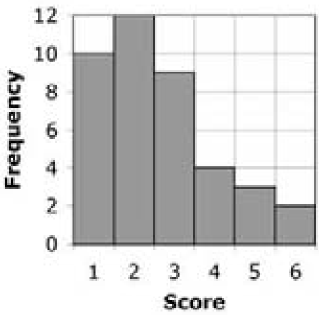


Pie chart

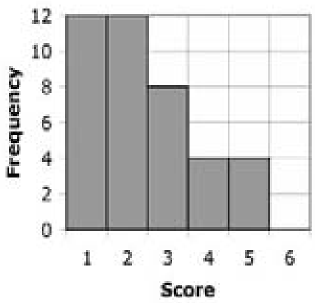


S5 Card set A – *Bar charts*

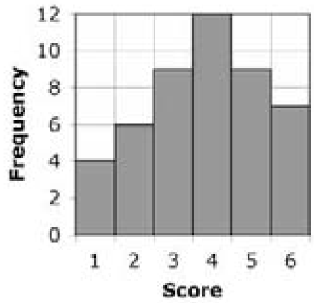
Bar chart A



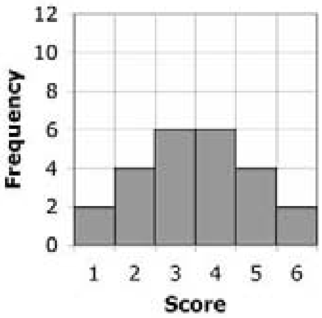
Bar chart B



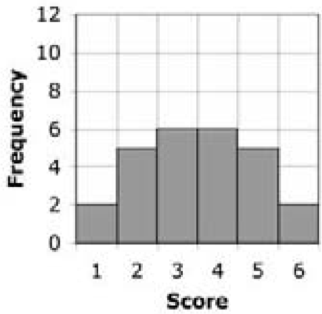
Bar chart C



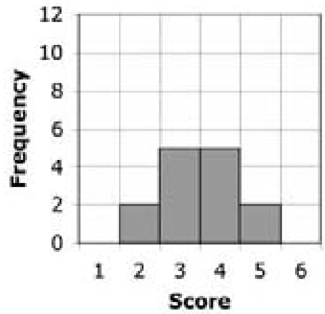
Bar chart D



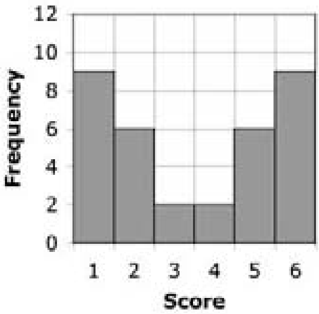
Bar chart E



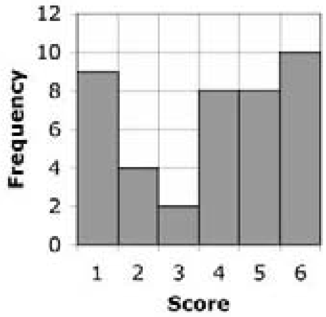
Bar chart F



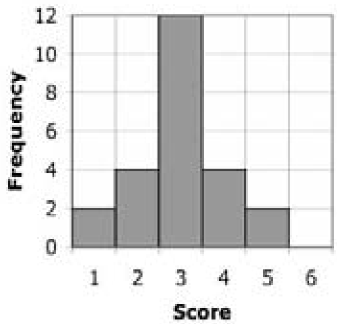
Bar chart G



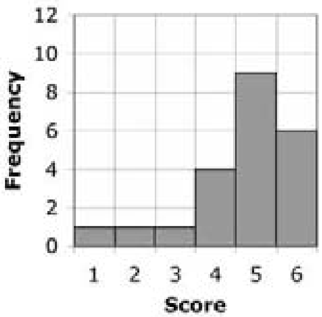
Bar chart H



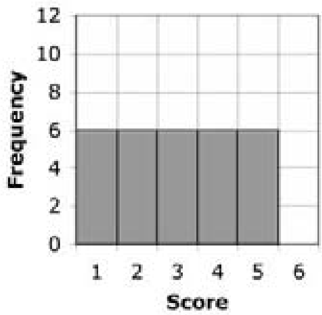
Bar chart I



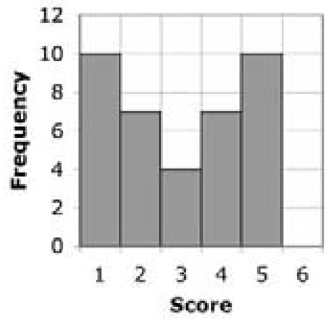
Bar chart J



Bar chart K

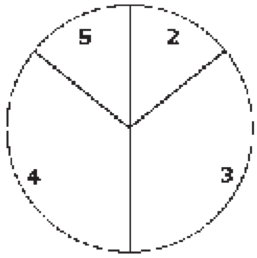


Bar chart L

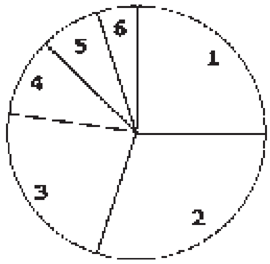


S5 Card set B – *Pie charts*

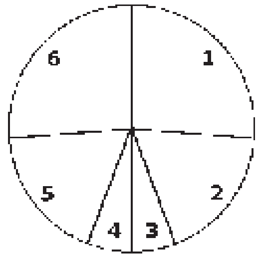
Pie chart A



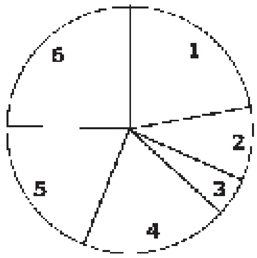
Pie chart B



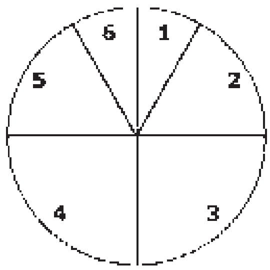
Pie chart C



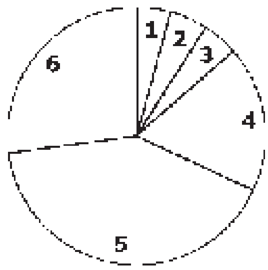
Pie chart D



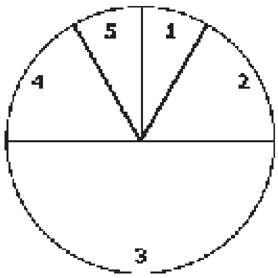
Pie chart E



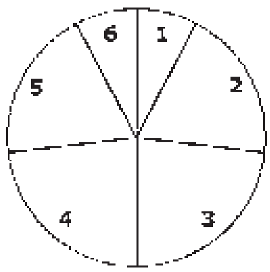
Pie chart F



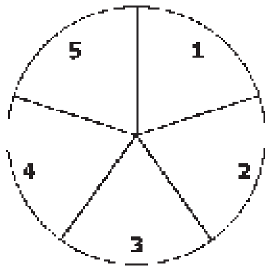
Pie chart G



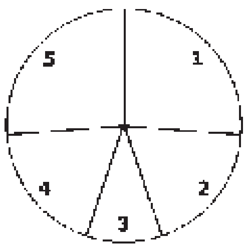
Pie chart H



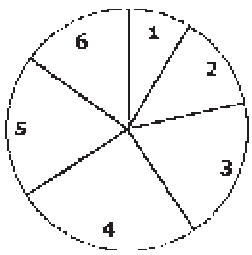
Pie chart I



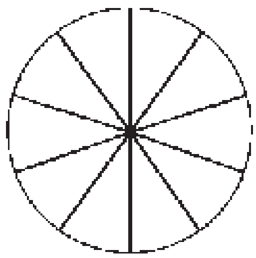
Pie chart J



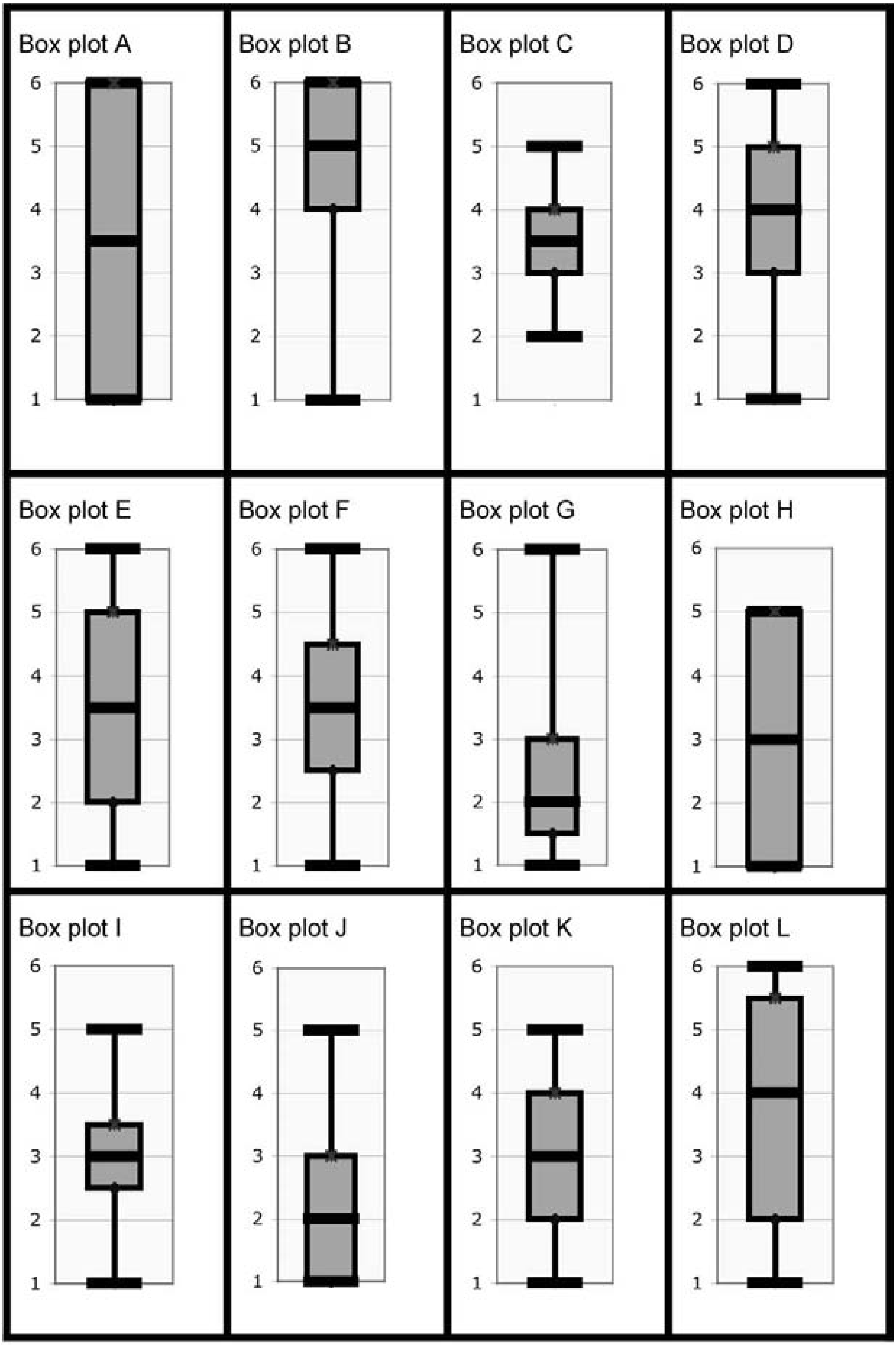
Pie chart K



Pie chart L
(complete this yourself)



S5 Card set C – *Box and whisker plots*



S5 Card set D – *Making your own cards*

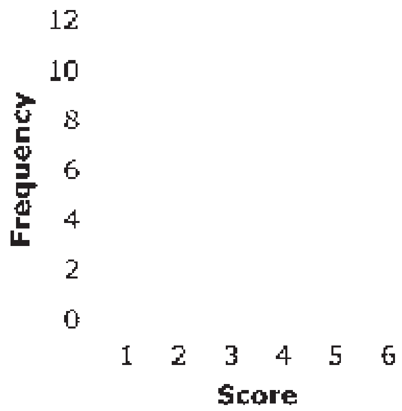
Frequency table

Score	1	2	3	4	5	6
Frequency						

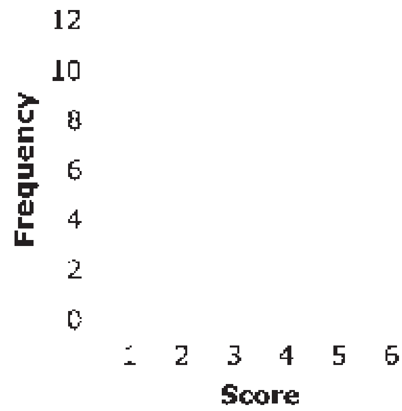
Frequency table

Score	1	2	3	4	5	6
Frequency						

Bar chart



Bar chart



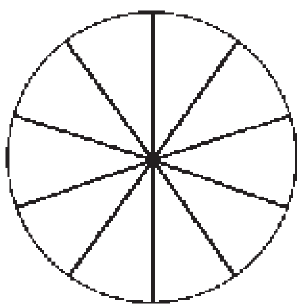
Statistics

Mean	
Median	
Mode	
Range	

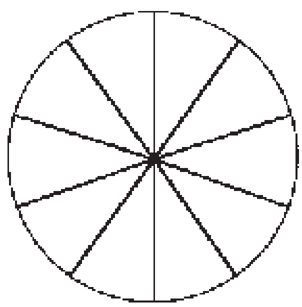
Statistics

Mean	
Median	
Mode	
Range	

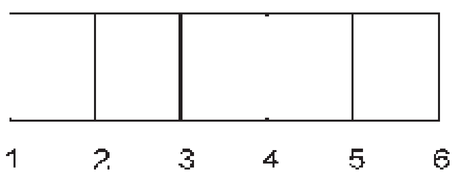
Pie chart



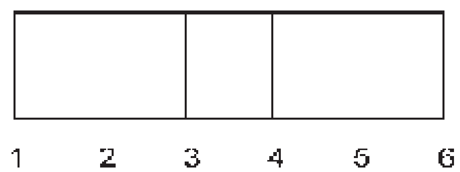
Pie chart



Box and whisker plot



Box and whisker plot



S6 • Interpreting frequency graphs, cumulative frequency graphs, box and whisker plots

Mathematical goals

To help learners to:

- interpret frequency graphs, cumulative frequency graphs, and box and whisker plots, all for large samples;
- see how a large number of data points can result in the graph being approximated by a continuous distribution.

Starting points

It is recommended that, before doing this session, learners try the similar but easier session **S5 Interpreting bar charts, pie charts, box and whisker plots**.

Materials required

- Overhead projector;
- OHT 1 – *Discrete and continuous representations*;
- OHT 2 – *Cumulative frequency, box and whisker*;
- OHT 3 – *Frequency, cumulative frequency, box and whisker.*

For each small group of learners you will need:

- Card set A – *Frequency graphs*;
- Card set B – *Statements*;
- Card set C – *Cumulative frequency graphs*;
- Card set D – *Box and whisker plots*;
- Sheet 1 – *Make up your own examples*;
- large sheet of paper for making a poster;
- felt tip pens;
- glue stick.

Time needed

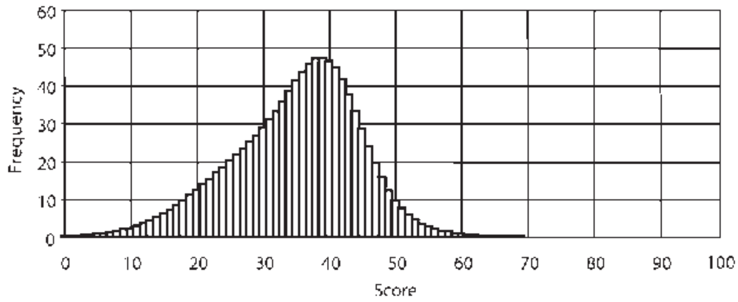
About 1–2 hours.

Suggested approach

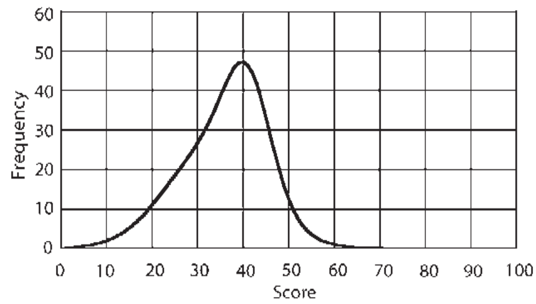
Beginning the session

Display OHT 1 – *Discrete and continuous representations*. Explain to learners that the bar chart represents the scores of candidates in an examination where the maximum mark was 100.

Frequency bar chart



Frequency line graph



When we have a distribution like this, where there are many bars close together, we can represent the graph as a continuous line, as in the frequency graph. This makes it a little easier to read off values.

What can you say about the exam? Did the candidates find it hard or easy? How can you tell?

What was the range of scores?

What was the modal score?

Do you think the mean score was the same as the mode, or was it higher or lower? How can you tell?

Roughly how many candidates sat the exam?

Could we begin to draw the frequency table for this data? It would take a long time, wouldn't it?

Explain that the grouped frequency table shows the same data in summary form. It is quite hard to see this, as the values are the sums of the heights of all the small bars.

Display OHT 2 – *Cumulative frequency, box and whisker.*

Score	0–10	11–20	21–30	31–40	41–50	51–60	61–70
Frequency	10	60	230	420	250	28	2

Score	≤10	≤20	≤30	≤40	≤50	≤60	≤70
Cumulative frequency	10	70	300	720	970	998	1 000

Explain that the cumulative frequency row displays a running total of the data. Ask the group to help you complete it.

Can you now see how many candidates sat the exam?
How many scored less than or equal to 50 marks?
How many scored more than half marks?

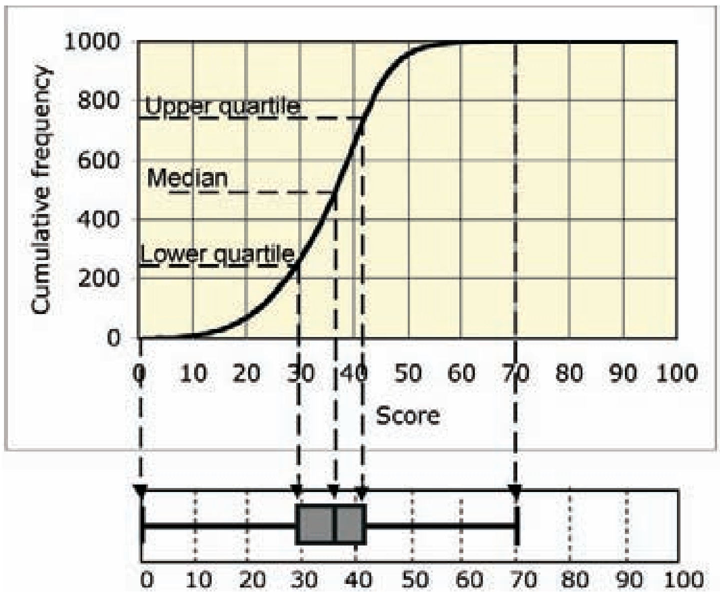
Explain that examiners often need to work out the median value and the quartiles of the distribution, so that they can compare the standard of the exam from one year to the next. After all, one would expect the 'average' student from one year to do about as well as the 'average' student the next year. This is easily done using a cumulative frequency diagram.

Plot the cumulative distribution with the group. This does not need to be done in detail.

The median score will be that obtained by the 500th candidate.
How can we find the median from this graph?

The lower quartile will be the score obtained by the 250th candidate.
How can we estimate this from the graph?

In this way, develop the box and whisker plot from the graph:



Now, introduce the task for the session, using OHT 3 – *Frequency, cumulative frequency, box and whisker.*

In this session, your job is to try to match the frequency graph, cumulative frequency graph, and box and whisker plot.

Working in groups

Give each pair of learners Card sets A, B, C and D. They should take it in turns to match the frequency graphs, the statements, the cumulative frequency graphs and the box plots. When they are certain they have a correct combination, they should stick them on a poster and write an explanation of how they know they match.

If learners have difficulty, stop the session and ask questions that may help to develop strategies. For example:

> If the cumulative frequency graph is flat between scores of 40 and 60, what does that tell you about the frequency graph between these scores?
>
> When the frequency graph is high, what does that say about the steepness of the cumulative frequency graph? Why?
>
> Can you see a link here with speed–time graphs and distance–time graphs?
>
> How can you get the range from a frequency graph? . . . from a cumulative frequency graph?

Reviewing and extending learning

Invite learners to come to the front and share what they have learned. Each group can be asked to present its poster and explain how they know that particular cards match.

What learners might do next

Learners who complete this task may like to make some new cards of their own. Sheet 1 – *Make up your own examples* has been provided for this purpose.

In addition, if learners wish to explore cumulative frequency graphs further, using a computer, there are programs on the internet that allow students to manipulate a cumulative frequency graph and observe what happens to the frequency bar chart and the box and whisker plot simultaneously.

Further ideas

The graphs and other ideas in this session may be transferred directly to the consideration of speed–time and distance–time graphs.

This activity uses multiple representations to deepen understanding of statistical ideas. This type of activity may be used in any topic where a range of representations are used. Examples in this pack include:

N1 Ordering fractions and decimals;

N5 Understanding the laws of arithmetic;

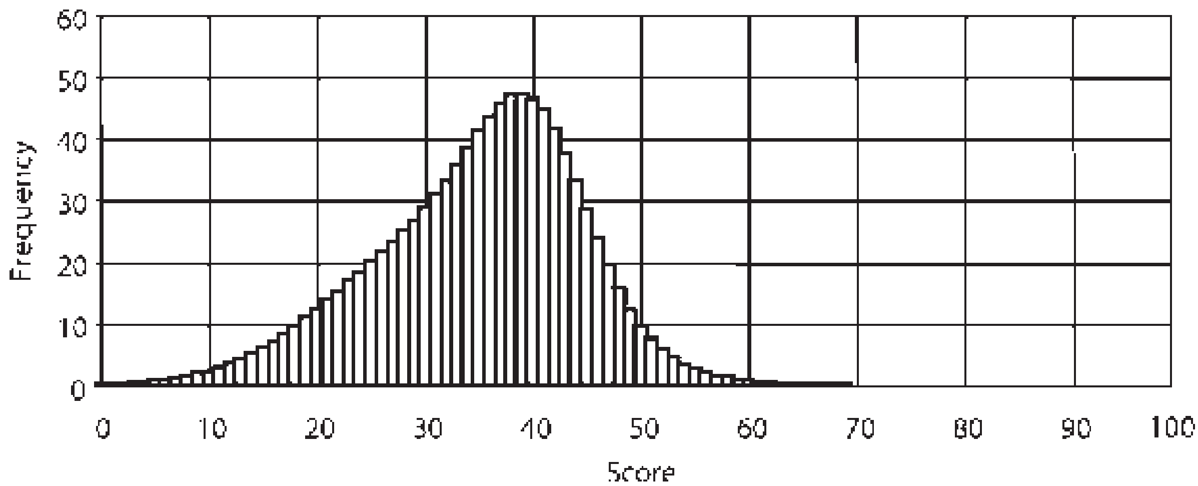
A1 Interpreting algebraic expressions;

A6 Interpreting distance–time graphs;

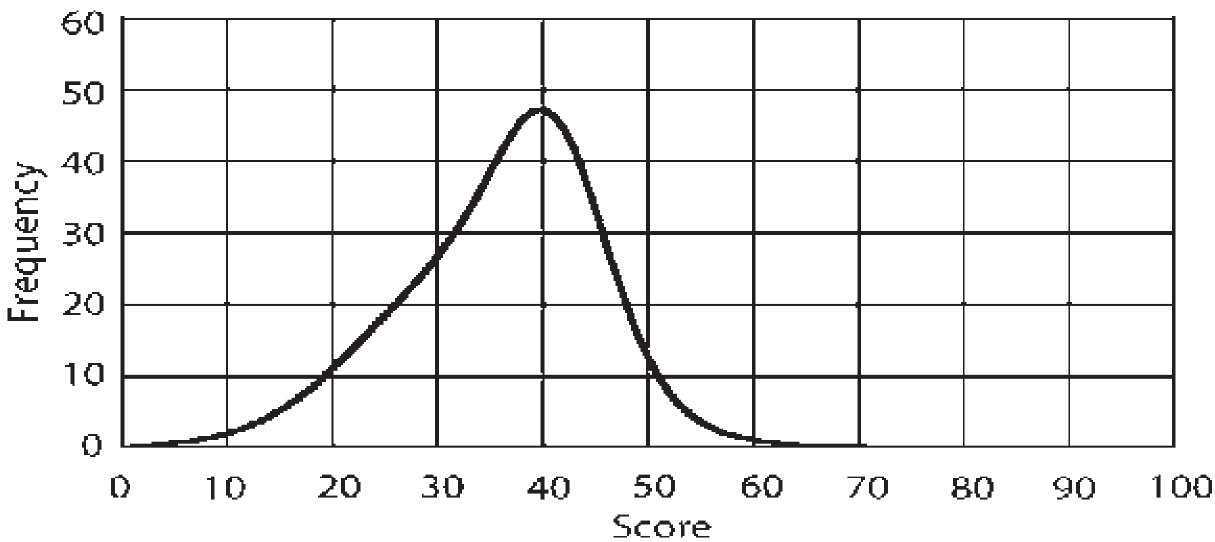
SS6 Representing 3D shapes.

S6 OHT 1 – *Discrete and continuous representations*

Frequency bar chart



Frequency line graph



Grouped frequency table

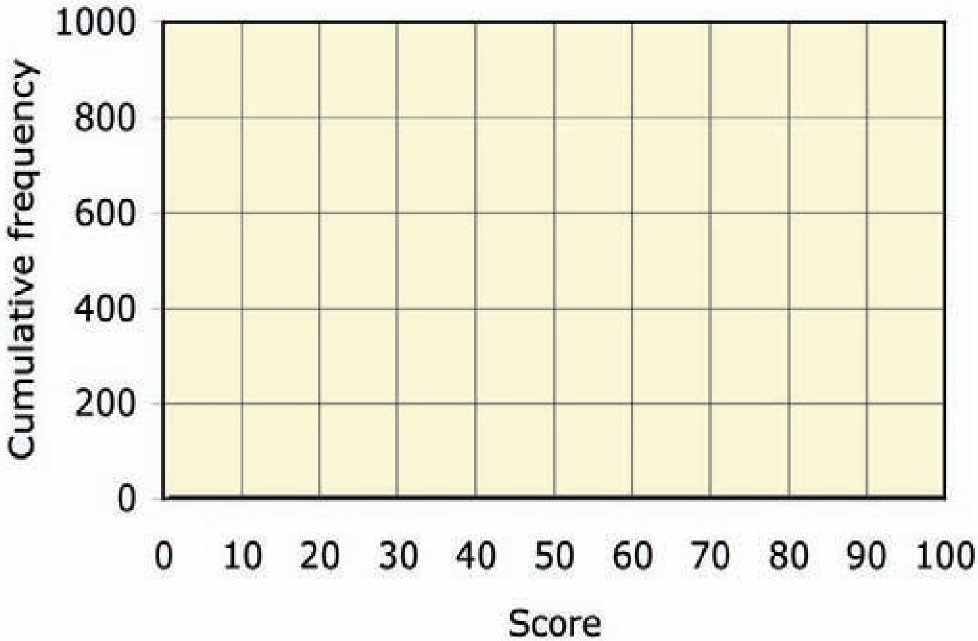
Score	0-10	11-20	21-30	31-40	41-50	51-60	61-70
Frequency	10	60	230	420	250	28	2

S6 OHT 2 – *Cumulative frequency, box and whisker*

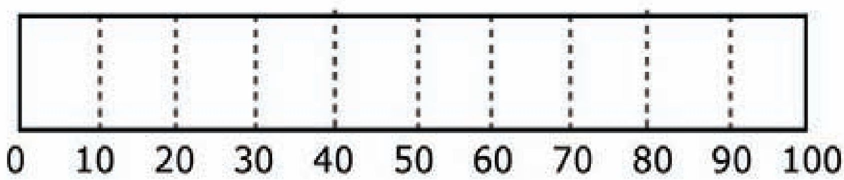
Score	0-10	11-20	21-30	31-40	41-50	51-60	61-70
Frequency	10	60	230	420	250	28	2

Score	≤ 10	≤ 20	≤30	≤ 40	≤ 50	≤ 60	≤ 70
Cumulative frequency							

Cumulative frequency graph

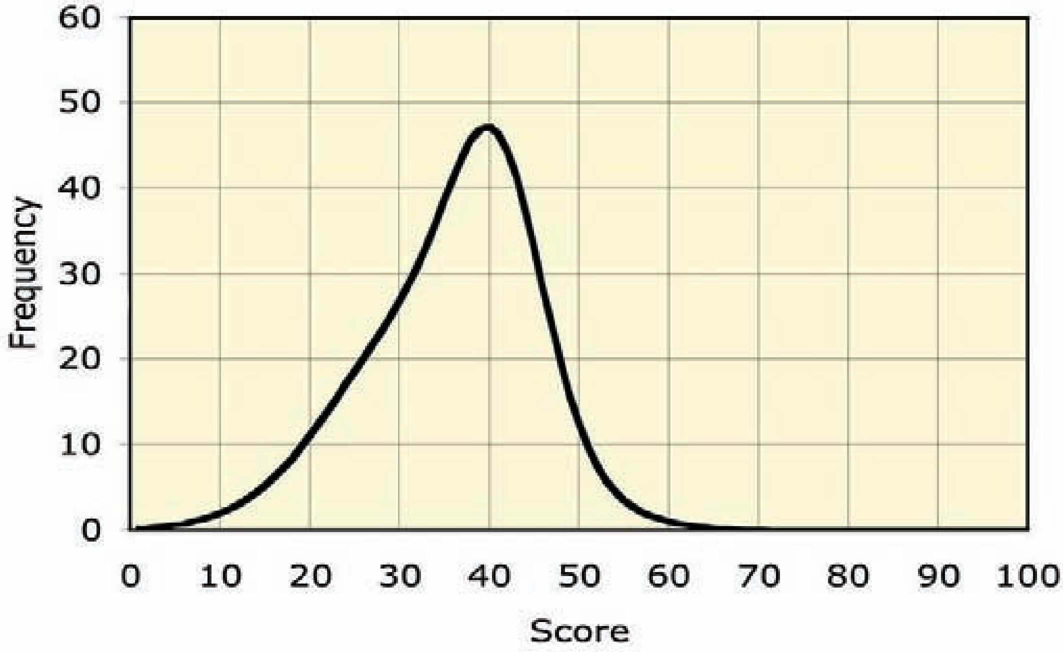


Box and whisker plot

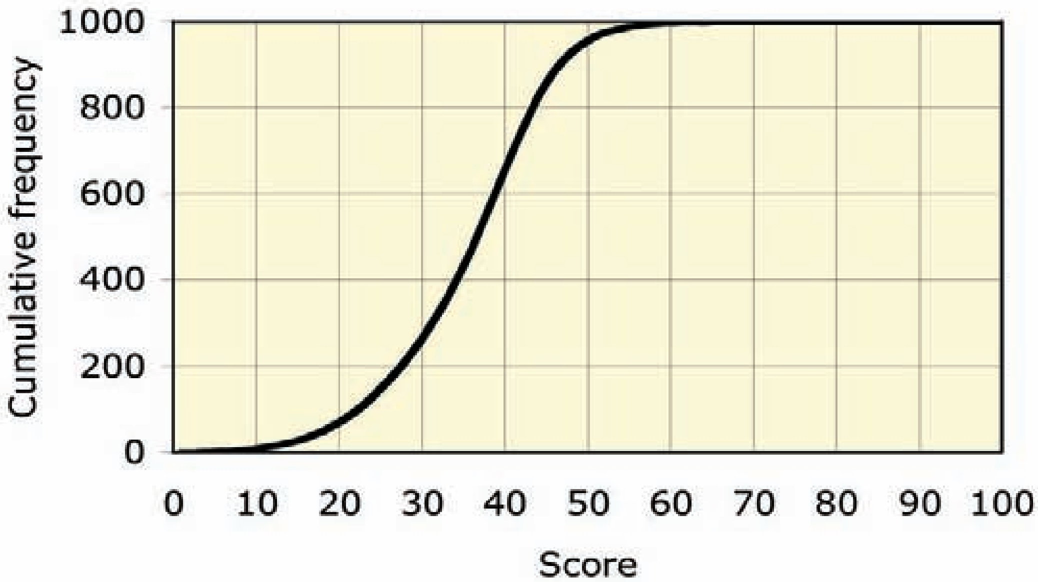


S6 OHT3 – *Frequency, cumulative frequency, box and whisker*

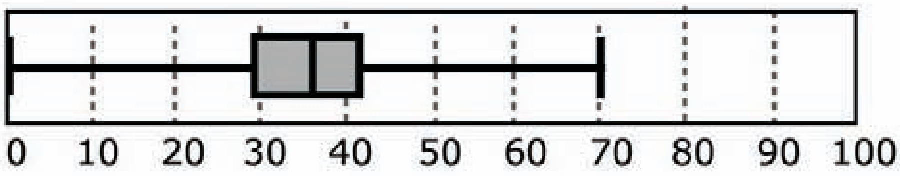
Frequency line graph



Cumulative frequency graph

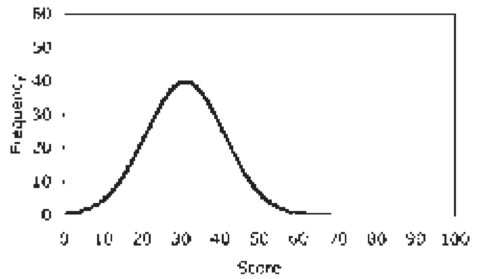


Box and whisker plot

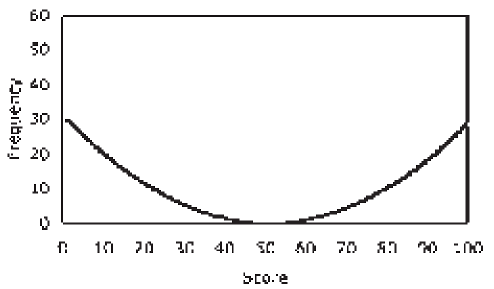


S6 Card set A – *Frequency graphs*

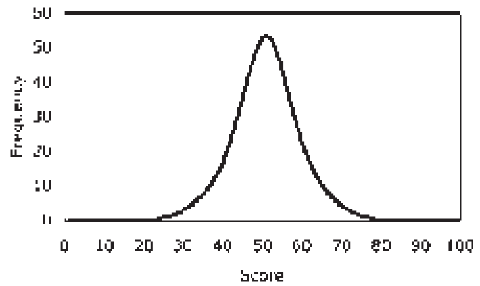
Frequency graph A



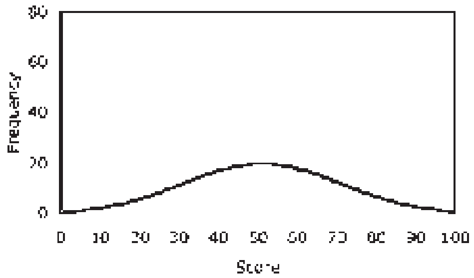
Frequency graph B



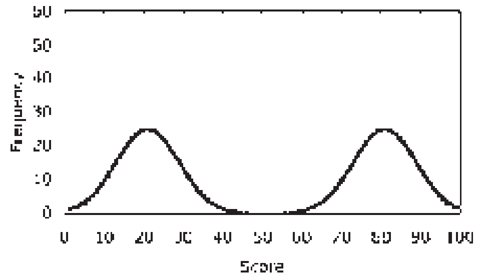
Frequency graph C



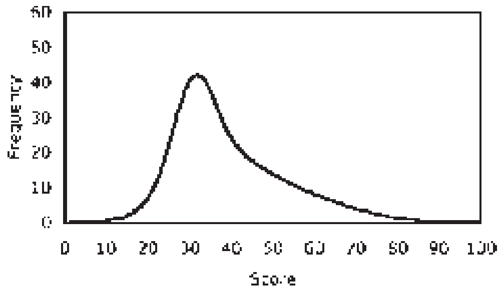
Frequency graph D



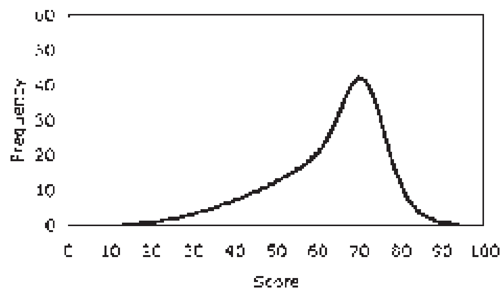
Frequency graph E



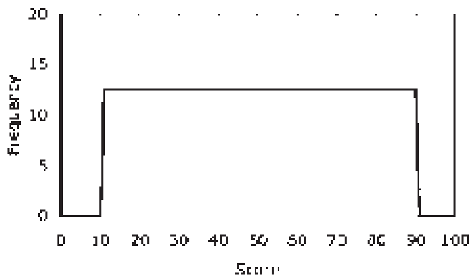
Frequency graph F



Frequency graph G



Frequency graph H



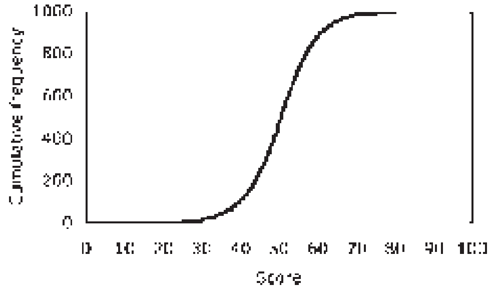
S6 Card set B – *Statements*

This was the sort of exam where you could either do everything or you couldn't get started.	**This exam did not sort out the better-performing candidates from the poorer-performing ones. They all got similar marks.**
Two groups of candidates sat the exam. One group had studied the work for two years. The other group had only just begun.	**This exam resulted in a huge spread of marks. Everyone could do something but nobody could do it all.**
In this exam, the mean mark was greater than the modal mark.	**In this exam, the mean mark was smaller than the modal mark.**
In this exam, the median and the modal marks were the same. There was a very wide range of marks.	**This exam was much too difficult for most candidates.**

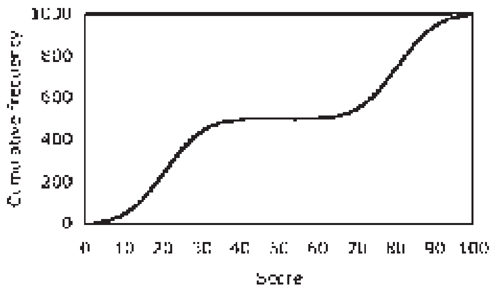
S6 Card set C – *Cumulative frequency graphs*

Cumulative frequency graph A

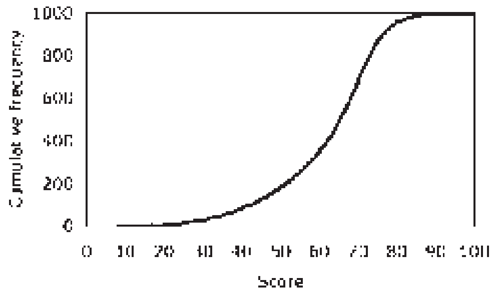
Cumulative frequency graph B



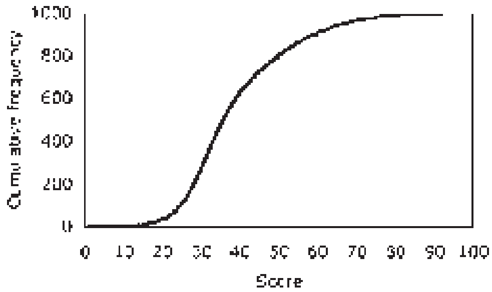
Cumulative frequency graph C



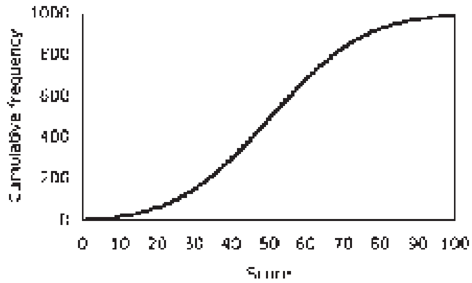
Cumulative frequency graph D



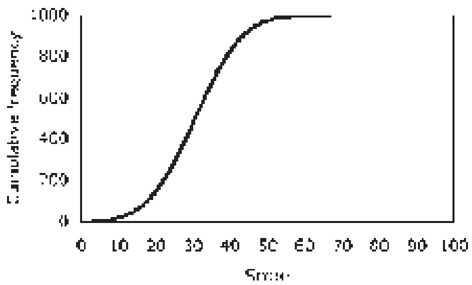
Cumulative frequency graph E



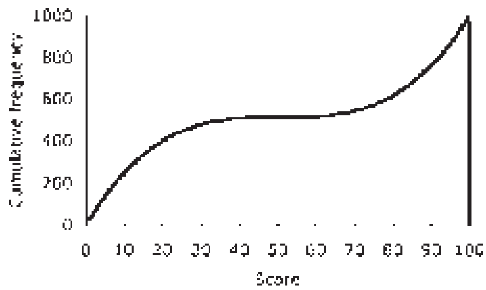
Cumulative frequency graph F



Cumulative frequency graph G

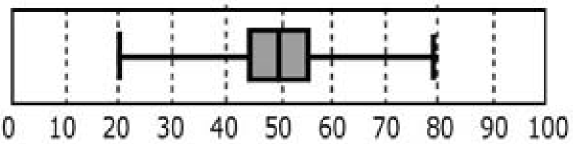


Cumulative frequency graph H

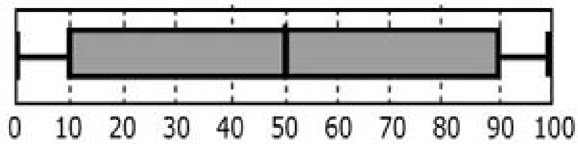


S6 Card set D – *Box and whisker plots*

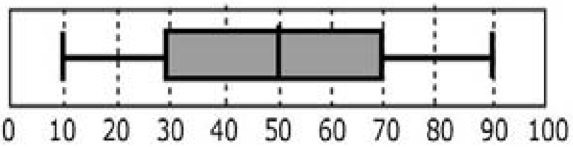
Box and whisker plot A



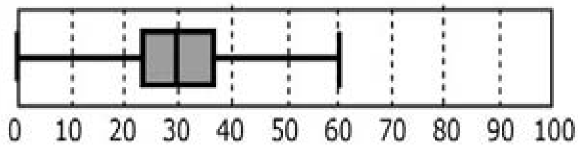
Box and whisker plot B



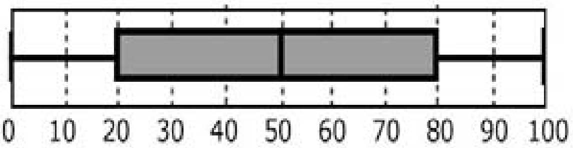
Box and whisker plot C



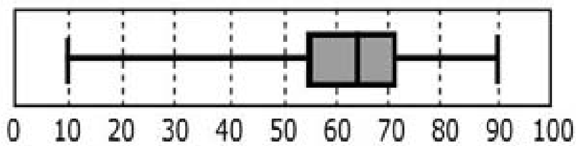
Box and whisker plot D



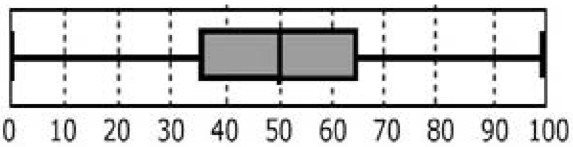
Box and whisker plot E



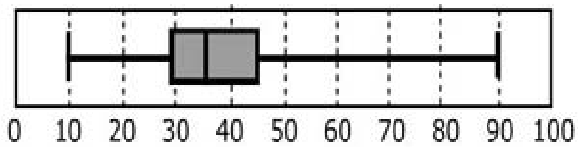
Box and whisker plot F



Box and whisker plot G

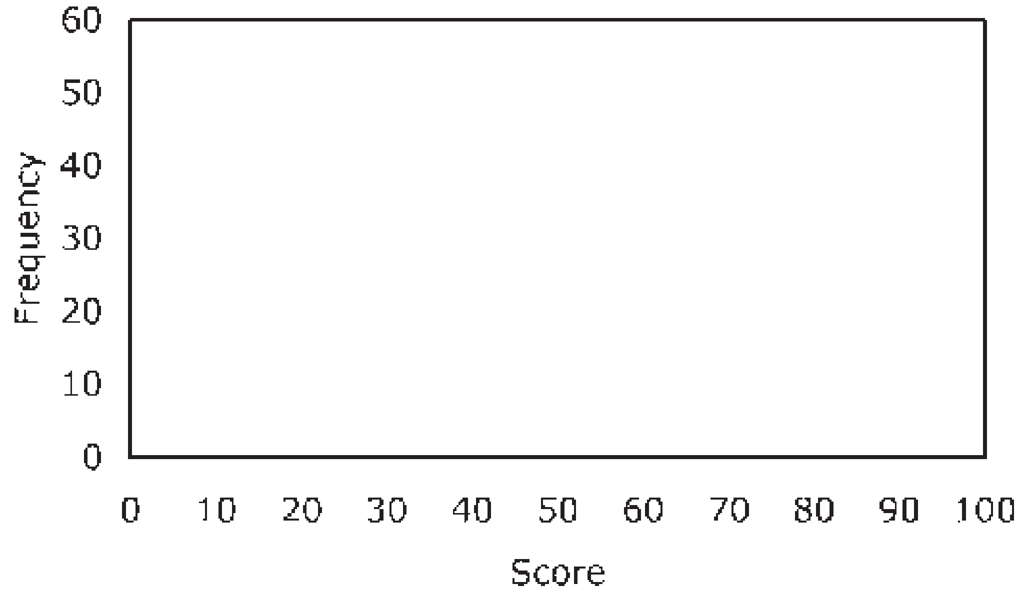


Box and whisker plot H

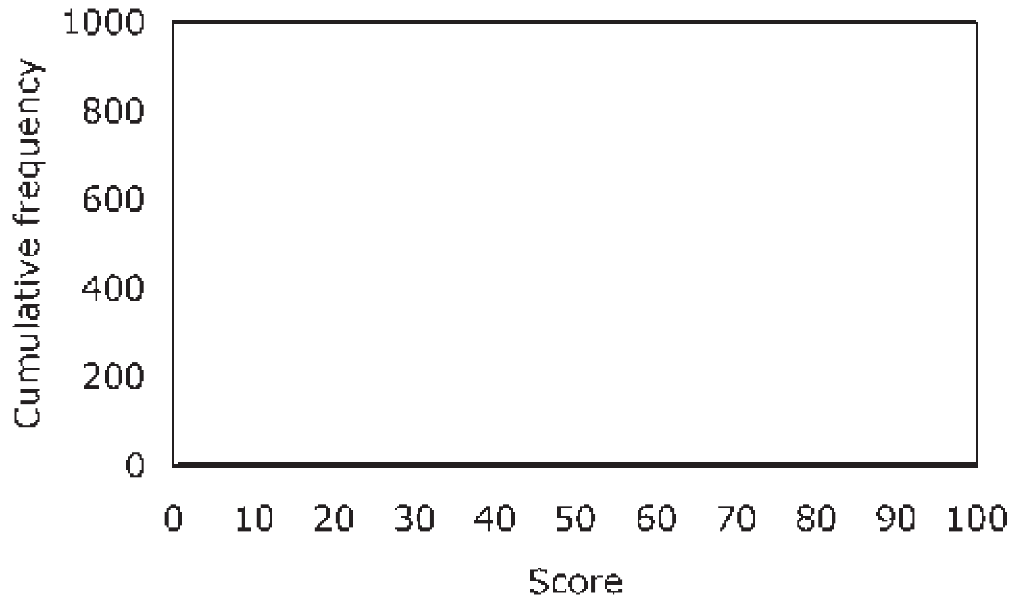


S6 Sheet 1 – *Make up your own examples*

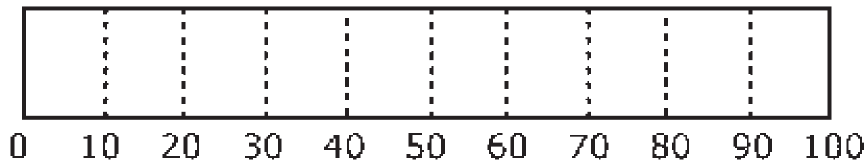
Frequency graph



Cumulative frequency graph



Box and whisker plot



S7 • Developing an exam question: probability

Mathematical goals

To help learners to:

- use past examination papers creatively;
- analyse the demands made by examination questions;
- understand and use estimates or measures of probability from theoretical models;
- list all outcomes for two successive events in a systematic way;
- identify mutually exclusive outcomes and know that the sum of the probabilities of all these outcomes is 1.

Starting points

Most learners will have attempted to answer probability questions such as these before, but they may have misconceptions. This session aims to expose learners' misconceptions and to enrich their understanding of the probability concepts and skills likely to be demanded in GCSE examination questions.

Materials required

- An OHT of Sheet 1 – *Spinners*.

For each learner you will need:

- Sheet 1 – *Spinners*;
- Sheet 2 – *Template for spinners question.*

Time needed

This session will take about an hour.

Suggested approach

Beginning the session

Ask learners to work in pairs on the GCSE examination question in Sheet 1 – *Spinners*. When everyone has had time to have a go at this, hold a whole group discussion on the approaches used.

Whole group discussion

(i) Completing the question

Place the OHT of Sheet 1 – *Spinners* on the overhead projector.

In order to answer this question, learners will need to recall that, when we have equally likely outcomes:

$$\text{The probability of an event} = \frac{\text{the number of ways the event can happen}}{\text{the total number of equally likely outcomes}}$$

Collect suggestions for correct answers to question 1 and write these numbers into the table on the OHT.

		Spinner A		
		1	**3**	**5**
Spinner B	**2**	3	5	7
	4	5	7	9
	5	6	8	10

After they have agreed on the correct values to be written in all the cells in the table, invite learners to suggest answers to question 2, explaining their reasoning. Discuss any disagreements.

> The probability that Sam wins a prize is $\frac{1}{3}$ because there are three players.
> The probability of Sam winning is $\frac{4}{9}$ because four of the nine numbers are 5 or 7.

Finally, consider the concept of mutually exclusive outcomes adding to 1.

> Why don't the three probabilities add up to 1?
> (Because, if the total is 6, Max and Amy will both win.)
> Yes, that is another way of saying that both events are not mutually exclusive. Max winning a prize does not exclude the possibility that Amy will also win a prize.

Whole group discussion

(ii) Generating further questions – same situation

There are many other questions an examiner might ask about this situation. Ask learners to suggest alternatives. In doing this, they should not seek to change the situation in any way, but simply ask new questions about the given situation.

For example, question 1 could have been:

> Draw a tree diagram to show the outcomes.

Question 2 could ask other probability questions such as:

> What is the probability that nobody wins a prize?
> What is the probability that only one person wins a prize?
> What is the probability that two people win a prize?

We could ask further questions where you can add probabilities:

> What is the probability of getting a multiple of 5 or a multiple of 4?

and where you cannot:

> What is the probability of getting a multiple of 5 or a multiple of 2?

or harder questions where you multiply probabilities:

> What is the probability that Amy wins two games in a row?

Working in groups (1)

Ask learners to choose questions they think they can answer and encourage them to work on it in pairs and try to agree on the correct answers. Learners may like to write and compare their different answers on the board.

Whole group discussion

(iii) Developing the situation

Hand out Sheet 2 – *Template for spinners questions*.

Ask learners to write their own GCSE question using this template.

Discuss with them how they might do this. They will need to draw two fair spinners, decide on the kinds of numbers they want on the spinners (e.g. negative numbers), how they may be combined (adding, multiplying, etc.), and what will happen for various different outcomes. They can choose whether to ask for a table or a tree diagram, and can ask any further questions.

If some learners feel more ambitious, they may like to try to develop spinners that are unfair, or use more than two spinners.

Working in groups (2)

The new questions should be passed around the group to be answered by other learners. Where learners have difficulty answering questions, the question-writers should explain what they intended and act as a teacher, helping other learners to answer the questions.

Alternatively, some of the new questions can be photocopied for further sessions or for homework.

Reviewing and extending learning

Finally, hold a whole group discussion about what has been learned, drawing out any continuing misconceptions. You should include a discussion of the level of difficulty of the new questions.

What learners might do next

Ask learners to choose another question from an exam paper and follow the process adopted in this session.

(i) Answer the question;

(ii) Ask new questions about the same situation (and answer them);

(iii) Change the situation and write a new question.

Further ideas

This method for developing exam questions can be used in any topic. Examples in this pack include:

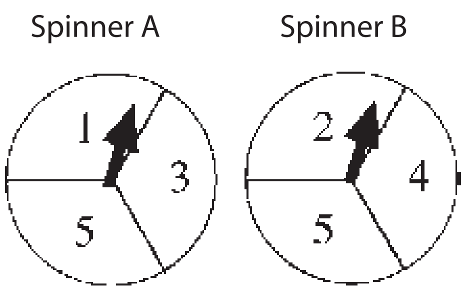
N10 Developing an exam question: number;

A8 Developing an exam question: generalising patterns;

SS8 Developing an exam question: transformations.

S7 Sheet 1 – *Spinners*

Two fair spinners are numbered 1, 3, 5 and 2, 4, 5 respectively.



You spin the spinners and add the numbers together.

- If the total is even, Amy wins a prize.
- If the total is a multiple of 3, Max wins a prize.
- If the total is 5 or 7, Sam wins a prize.

1. Draw a table to show all the equally likely outcomes:

		Spinner A		
		1	**3**	**5**
Spinner B	**2**			
	4			
	5			

2. Complete the table below:

Name	Probability of winning a prize
Amy	$\frac{1}{3}$
Max	
Sam	

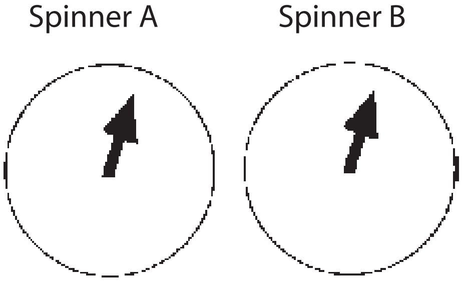
3. Explain what is wrong with the following statement:

   "The probability that someone will win is 1. This means that the probabilities in question 2 should add up to 1."

4. What further questions can you ask?

S7 Sheet 2 – *Template for spinners question*

Two fair spinners are numbered .... and .... respectively.



You spin the spinners and .... the numbers together.

If the result is .... then ....

If the result is .... then ....

If the result is .... then ....

1. Draw a .... to show all the equally likely outcomes:

2. ....

3. ....

S8 • Using binomial probabilities

Mathematical goals

To enable learners to:

- calculate binomial probabilities;
- calculate cumulative binomial probabilities.

To develop learners' understanding of:

- the context in which it is appropriate to use binomial probabilities;
- the symmetrical nature of the formula for a binomial probability, e.g. P(2 right out of 13) = P(11 wrong out of 13);
- alternative strategies for calculating cumulative binomial probabilities, e.g. P(at least 3 successes out of 10 trials) = 1 – P(fewer than 3 successes).

Starting points

Learners should understand what is meant by independent events and how to calculate probabilities of a series of independent events using $P(A \text{ and } B) = P(A) \times P(B)$.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Events*;
- Card set B – *Probabilities*.

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Discuss what independent events are and how to calculate the probability of successive independent events.

Whole group discussion (1)

Use successively larger tree diagrams to establish the formula for binomial probabilities. Introduce the notation $\binom{12}{9}$ or ${}^{12}C_9$ and check that learners can use their calculators to evaluate it. Emphasise the criteria that must be true in order for the binomial to be applied. Ask learners to come up with some repeated events that are clearly not suitable for use with binomial probabilities e.g. the probability that it rains at midday and on successive days.

Give each learner a binomial probability such as ${}^{15}C_7\ 0.8^7 \times 0.2^8$ and ask them to write a question for which that is the answer. Collect in the questions and read them out one at a time. Ask learners to write the answer to each question on their whiteboards and compare these with the original answer. Discuss any differences and, in particular, any questions in which the information is not clear enough in order to arrive at the answer. Check that the events given in each question are appropriate for use with binomial probabilities.

Working in groups

Arrange learners in pairs and give each pair Card set A – *Events*. Set the scenario:

> "The probability of winning a game is always 0.6 and there are 8 games left to play."

Ask learners to match any events on the cards that are essentially the same event, e.g. P(lose at least 5) and P(win fewer than 4). Ask pairs of learners to compare with other pairs how many matchings they have until there is a consensus in the group as a whole.

Encourage learners who find the language difficult to draw a line and highlight the events that the card refers to e.g.

Losses	8	7	6	5	4	3	2	1	0
Wins	0	1	2	3	4	5	6	7	8

This example represents P(win 3 or more) or P(lose 5 or less).

If learners find this easy, ask them to add some more phrases for the events in each group, e.g. P(win 2 or less) can be placed with P(win fewer than 3).

Give out Card set B – *Probabilities*. Ask learners to match the probabilities with their sets of events from Card set A.

Whole group discussion (2)

Where there is more than one *Probabilities* card that matches a set of events, discuss the reason for this. Emphasise that it does not matter whether you consider the number of wins or the number of losses; the formulae give the same value.

Ask learners to find two events that, together, cover all possibilities, e.g. P(win more than 3) and P(win 3 or less). Discuss the possibility of calculating P(win more than 3) as 1 – P(win 3 or less) as a more efficient strategy.

Ask learners to identify other events with long probabilities that can be reduced in this way. Encourage learners to find the cards that contain these long probabilities and, for each one, write the alternative calculation on the card.

Reviewing and extending learning

Building on Whole group discussion (1), give each pair of learners a cumulative binomial probability and ask them to write a suitable question for which it is the answer. As before, collect in the questions and read them out one at a time. Learners write down the answer to the question on their mini-whiteboards. These are then compared with the original answer. Discuss any differences and in particular any questions in which the information is not clear enough in order to give the answer. Check that the events given in each question are appropriate for use with cumulative binomial probabilities.

What learners might do next

Use cumulative binomial tables for a larger number of repeated events.

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S8 Card set A – *Events*

P(win fewer than 3)	P(win no more than 3)
P(win at least 3)	P(lose more than 5)
P(win exactly 3)	P(win more than 3)
P(lose no more than 5)	P(lose at least 5)
P(lose exactly 5)	P(win fewer than 4)
P(lose fewer than 5)	P(lose more than 4)
P(lose 5 or less)	P(win 3 or more)
P(lose 5 or more)	P(win 3 or less)

S8 Card set B – *Probabilities*

${}^8C_5 0.4^5 \times 0.6^3 + {}^8C_4 0.4^4 \times 0.6^4 + {}^8C_3 0.4^3 \times 0.6^5 + {}^8C_2 0.4^2 \times 0.6^6 + {}^8C_1 0.4 \times 0.6^7 + 0.6^8$
${}^8C_2 0.6^2 \times 0.4^6 - {}^8C_1 0.6 \times 0.4^7 + 0.4^8$
${}^8C_6 0.4^6 \times 0.6^2 + {}^8C_7 0.4^7 \times 0.6 + 0.4^8$
${}^8C_3 0.6^3 \times 0.4^5 + {}^8C_4 0.6^4 \times 0.4^4 + {}^8C_5 0.6^5 \times 0.4^3 + {}^8C_6 0.6^6 \times 0.4^2 + {}^8C_7 0.6^7 \times 0.4 + 0.6^8$
${}^8C_4 0.4^4 \times 0.6^4 + {}^8C_3 0.4^3 \times 0.6^5 + {}^8C_2 0.4^2 \times 0.6^6 + {}^8C_1 0.4 \times 0.6^7 + 0.6^8$
${}^8C_3 0.6^3 \times 0.4^5 + {}^8C_2 0.6^2 \times 0.4^6 + {}^8C_1 0.6 \times 0.4^7 + 0.4^8$
${}^8C_5 0.4^5 \times 0.6^3 + {}^8C_6 0.4^6 \times 0.6^2 + {}^8C_7 0.4^7 \times 0.6 + 0.4^8$
${}^8C_4 0.6^4 \times 0.4^4 + {}^8C_5 0.6^5 \times 0.4^3 + {}^8C_6 0.6^6 \times 0.4^2 + {}^8C_7 0.6^7 \times 0.4 + 0.6^8$
${}^8C_3 0.6^3 \times 0.4^5$
${}^8C_5 0.4^5 \times 0.6^3$

C1 • Linking the properties and forms of quadratic functions

Mathematical goals

To enable learners to:

- identify different forms and properties of quadratic functions;
- connect quadratic functions with their graphs and properties, including intersections with axes, maxima and minima.

For learners working towards a lower level qualification, you can focus on the graphs and their intersections with the axes.

Starting points

Learners will need to be familiar with the following forms of quadratic functions and their interpretation:

$y = ax^2 + bx + c$,
$y = (x + a)(x + b)$,
$y = a(x + b)^2 + c$.

Learners working towards a lower level qualification will not need to know the form $y = a(x + b)^2 + c$

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Properties of quadratic graphs* (two pages);
- Card set B – *Graphs of quadratic functions*;
- large sheet of paper for making a poster;
- glue stick;
- felt tip pens;

and possibly

- access to computer with graph-drawing software;
- graphic calculators.

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Explain that there are seven sets of cards. Each set is made up of a quadratic function written in different forms, some associated properties, and a graph.

Learners working towards a lower level qualification should leave out cards in the form $y = a(x + b)^2 + c$ and those that refer to maxima or minima.

Working in groups

Ask learners to work in pairs. Issue Card set A and Card set B to each pair. Invite each pair to match cards into the seven sets. As they do this, encourage them to explain their reasoning both to you, as you move round the room, and to each other.

Learners may like to create posters by sticking a set of cards onto a large sheet of paper and writing their reasoning around each card.

If some learners are likely to find the task easy, you may wish to give them reduced sets of cards and ask them to create the missing ones. For example, give them Card set A without the $y =$. . . cards and ask them to create their own.

Learners who struggle may be helped by using some graph-drawing software or graphic calculators. They could focus on completing four or five sets rather than all seven.

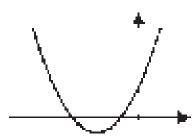
Learners who finish quickly may enjoy devising their own sets of cards at a more challenging level, e.g. when the coefficient of x^2 is not ±1.

Reviewing and extending learning

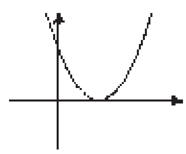
Ask a representative from each pair of learners to come to the board and explain how they decided that a set of cards belonged together.

Ask learners questions using mini-whiteboards. For example:

Give me a possible equation for this graph:



Give me a possible equation for this graph:



What are the x and y intercepts of $y = (x + 3)(x – 2)$? How can you tell?

What are the x and y intercepts of $y = (x + 3)^2$? How can you tell?

Show me the equation of a quadratic that intersects the y axis at –10. Now show me the same equation in a different form.

Show me the equation of a quadratic that intersects the x axis at –5 and +7. Now show me the same equation in a different form.

Where are the intercepts and where is the minimum of the function $y = (x - 4)^2 - 9$. How can you tell?

Show me the equation of a quadratic function with a minimum at (4, 6). Now show me the same equation in a different form.

Ask learners to produce generalisations of these results, such as:

Show me the equation of a quadratic with a minimum at (a, b). Now show me the same equation in a different form.

Learners working towards a lower level qualification should focus on the earlier questions.

Show me the equation of a quadratic with x intercepts at $(a, 0)$ and $(b, 0)$. Now show me the same equation in a different form.

What learners might do next

Learners could be asked to produce a graph from a given quadratic function, with intercepts and stationary points marked. Different levels of challenge could be available for learners to choose from, e.g. quadratics with a coefficient of x^2 that is not ±1.

Completed square form could be linked to translations of the graphs.

Finding stationary points using calculus could be introduced and linked with the completed square form of a quadratic function, to show that they both give the same result.

Further ideas

This idea could be extended and used with other types of functions, e.g. linear or trigonometric.

This session could be used to introduce the idea of quadratic functions. Learners would need access to a computer with graph-drawing software or a graphic calculator. They could use the computer or calculator to match up the properties with their functions and then work out for themselves how the different forms and properties link together.

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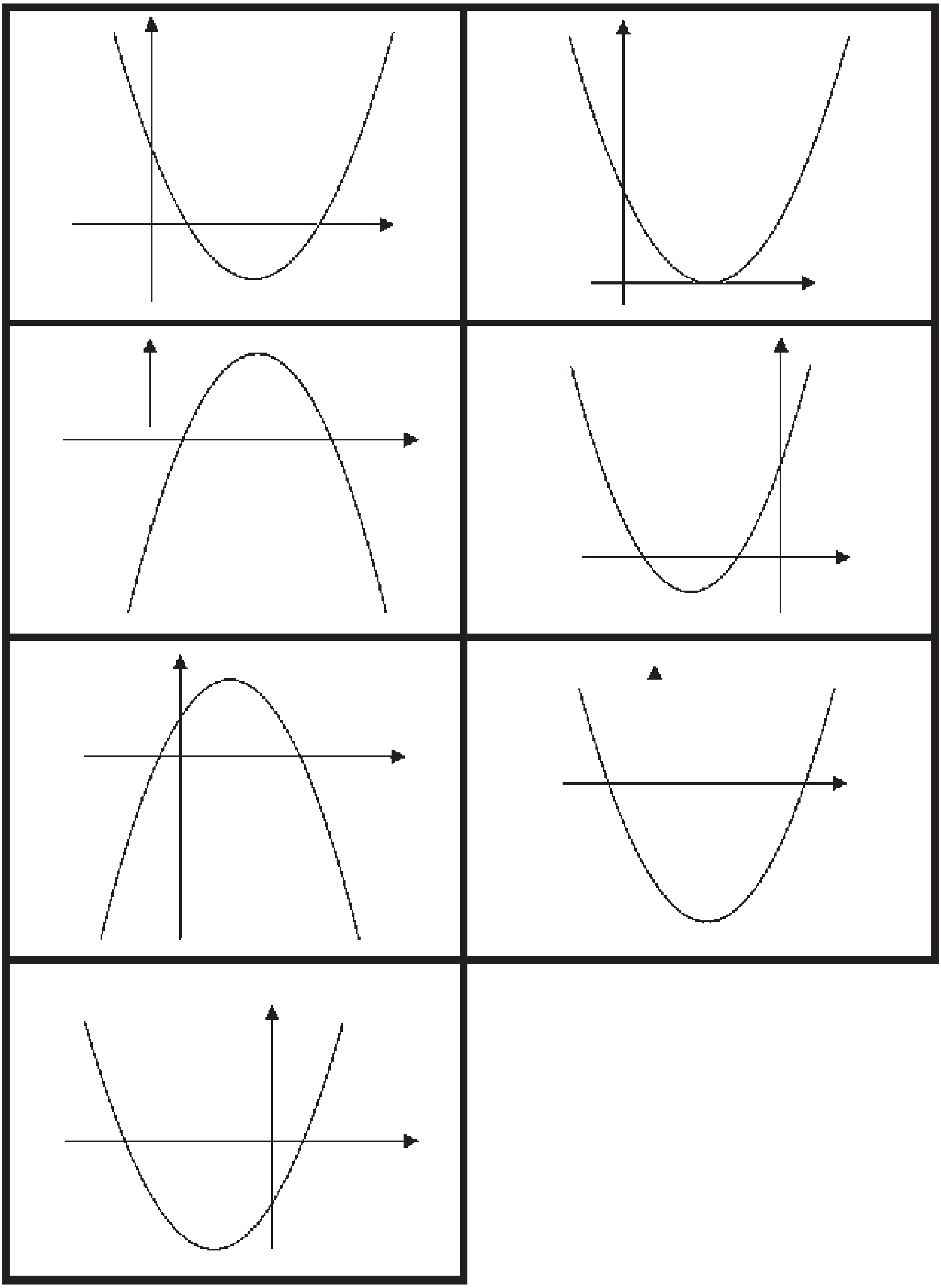
C1 Card set A – *Properties of quadratic graphs* (page 1)

$y = x^2 + 6x - 16$	$y = x^2 - 8x + 16$
$y = 8 - x^2 + 2x$	$y = 6x - x^2 - 8$
$y = x^2 - 10x + 16$	$y = x^2 + 6x + 8$
$y = x^2 - 6x - 16$	$y = (x - 8)(x + 2)$
$y = (x + 4)(x + 2)$	$y = (x + 2)(4 - x)$
$y = (x - 4)(2 - x)$	$y = (x - 8)(x - 2)$
$y = (x - 4)(x - 4)$	$y = (x + 8)(x - 2)$
$y = (x + 3)^2 - 25$	$y = (x - 4)^2$
$y = (x - 5)^2 - 9$	$y = -(x - 3)^2 + 1$
$y = -(x - 1)^2 + 9$	$y = (x + 3)^2 - 1$
$y = (x - 3)^2 - 25$	**Minimum at (3, –25)**
Minimum at (–3, –1)	**Maximum at (1, 9)**

C1 Card set A – *Properties of quadratic graphs* (page 2)

Maximum at (3, 1)	**Minimum at (5, –9)**
Minimum at (4, 0)	**Minimum at (–3, –25)**
$x = 0, y = -16$	$x = 0, y = 16$
$x = 0, y = 16$	$x = 0, y = -8$
$x = 0, y = 8$	$x = 0, y = 8$
$x = 0, y = -16$	$y = 0, x = 8 \text{ or } -2$
$y = 0, x = -4 \text{ or } -2$	$y = 0, x = -2 \text{ or } 4$
$y = 0, x = 4 \text{ or } 2$	$y = 0, x = 8 \text{ or } 2$
$y = 0, x = 4$	$y = 0, x = -8 \text{ or } 2$

C1 Card set B – *Graphs of quadratic functions*



C2 • Exploring functions involving fractional and negative powers of x

Mathematical goals

To develop learners' ability to:

- find the stationary points of a function and determine their nature;
- solve appropriate equations in order to find the intercepts of a function.

To encourage learners to:

- connect the mathematical properties of a function and relate them to the graph.

Starting points

Learners should understand that $f'(x) = 0$ finds stationary points and $f''(x)$ determines their nature. Learners should also be familiar with fractional and negative indices.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- at least one large sheet of paper for making a poster;
- felt tip pens;
- graphic calculator (optional).

Time needed

At least 45 minutes.

Suggested approach

Beginning the session

Use mini-whiteboards to do a quick revision of differentiating functions with fractional and negative indices such as:

$$y = \sqrt{x} \qquad y = \frac{1}{x} \qquad y = \frac{3}{x^2} \qquad y = \frac{1}{2\sqrt{x}}$$

Working in groups (1)

Write the following list of functions on the board.

Set A	Set B	Set C
$f(x) = x^2 + \frac{8}{x}$	$f(x) = x + \frac{2}{x} + 3$	$f(x) = \sqrt{x} - 3 + \frac{2}{\sqrt{x}}$
$f(x) = x + \frac{1}{x^2}$	$f(x) = \sqrt{x} - x$	$f(x) = 1 - \frac{1}{x} - \frac{2}{x^2}$
$f(x) = x + \frac{8}{x^2}$	$f(x) = 2\sqrt{x} - x$	$f(x) = \sqrt{x} - x^2$
$f(x) = x^2 - \frac{8}{x}$		
$f(x) = x^2 + \frac{1}{x}$		

Ask learners to work in pairs and ask each pair to choose a function. Explain that set A are normal challenge, set B are medium challenge and set C are high challenge. The extra challenge tends to come from the equations that need to be solved and manipulated.

Ask them to explore their chosen function (i.e. find out all they can about the intercepts, stationary points, and their nature) and then draw a good quality sketch of the function based on this information.

Explain that all their rough work, working out and thinking must be written on the large sheet of paper, which will become a poster.

Remind learners that crossings out and mistakes are all part of the learning.

As the pairs are working on their functions, go round and ask them to talk you through what they are doing. Note on the board any problems or interesting points that arise, for use in whole group discussion at the end of the session.

If learners finish early, they can check their graph on a graphic calculator or computer and consider what happens to the shape of the graph as x approaches infinity and justify why this is so from the equation.

Reviewing and extending learning

When all pairs have found sufficient information to sketch their graph, ask them to show their poster and share the findings. Ask each pair to talk about something that they found either difficult or interesting. Also, discuss any points that you have noted on the board.

What learners might do next

Learners could look at integrating these functions. They could then link integrals with areas using limits. Relating these to their graphs will help them to realise that, because of the breaks in the graph, not all limits are possible.

Further ideas

This idea can be applied to any type of function, e.g. quadratics and cubics.

It can also be used to investigate the effects of transformations on a function rather than stationary points and intercepts, e.g. trigonometrical or modulus functions.

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C3 • Matching functions and derivatives

Mathematical goals

To enable learners to:

- practise differentiating quadratic functions;
- practise finding the values of a function and its derivative at specific points;
- distinguish between $f(x)$ and $f'(x)$;
- relate values of $f(x)$ and $f'(x)$ to the graph of $y = f(x)$;

and to reflect on and discuss these processes.

Starting points

Learners should have some understanding of:

- function notation;
- differentiation of quadratic functions.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Functions*.

Time needed

At least 30 minutes.

Suggested approach

Beginning the session

Explain to learners that they will be given a set of 20 cards relating to five different functions. Learners have to make five groups of cards, so that each group refers to the same function.

Working in groups

Ask learners to work in pairs. Give each pair a set of cards.

Some learners may be unclear about the difference between, for example, $f(2)$ and $f'(2)$. If this is the case, invite other learners to explain.

Learners who are likely to struggle may be given a smaller set of cards and asked to create only three or four groups.

Learners who are likely to find the task easy may be given a set of cards with some of the $f(x) = \ldots$ cards missing. Learners have to create these cards to make complete groups.

Learners who finish quickly may be asked to make supplementary cards to add to each group, or they may enjoy devising their own groups at a more challenging level. These may then be given to other learners.

When most pairs have finished and every pair has created at least three groups of cards, ask learners, either in their pairs or in groups of four, to discuss what each card tells them about the graph of that function.

Ask each pair to explain to the whole group why they matched one of their cards to a particular function and what it tells them about the graph.

Reviewing and extending learning

Using the mini-whiteboards, ask learners questions such as the following:

If $f(x) = 3x^2 - 2x - 1$, then show me the value of $f(0)$, $f(3)$, $f'(0)$, $f'(3)$.

Show me a function where $f(2) = 1$.

Show me a function where $f'(2) = 1$.

If $f'(x) = 2x - 3$, then show me a possible function $f(x)$.

Sketch me a graph of a function where $f(0) = 10$.

Then sketch one where $f(3) = 0$.

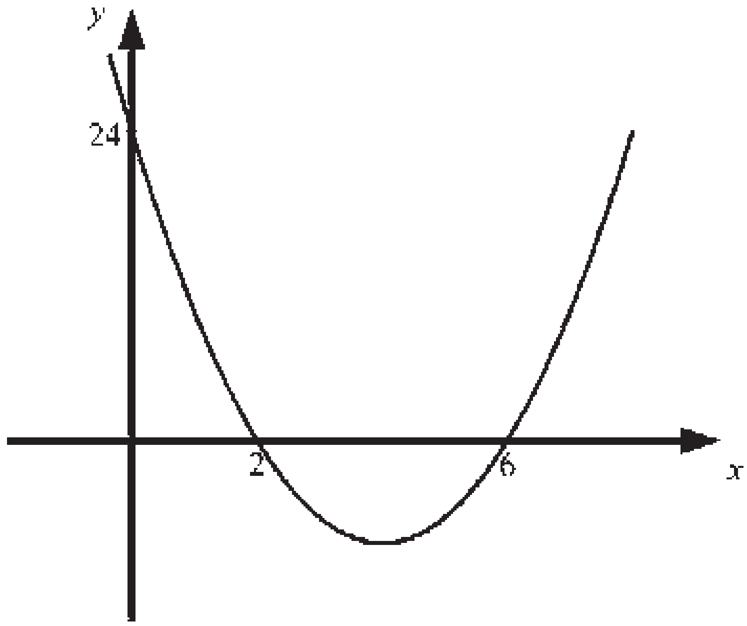
Sketch a graph of a function where $f'(0) = 0$.

Sketch a graph of a function where $f'(2) = 0$.

Sketch a graph of a function where $f'(0) = 3$.

What learners might do next

Ask learners to write an explanation of the difference between $f(4)$ and $f'(4)$ for a function of their own choice.



Ask learners to describe as much as they can about $f(x)$ and $f'(x)$ at various points on this graph.

This session could be developed into work with stationary points.

Further ideas

This idea could be used with other types of functions, e.g. cubic functions, or functions with fractional or negative indices.

This session could be adapted to practise substitution skills using surds, negative numbers or fractions.

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C3 Card set A – *Functions*

$f(x) = 3x^2 - 3x + 1$	$f(3) = -3$
$f(x) = x^2 - 12$	$f(0) = 5$
$f(x) = 2x - x^2 + 3$	$f(-1) = 0$
$f(x) = 4x^2 - 6x + 5$	$f(1) = 3$
$f(x) = 4x^2 - 3x + 2$	$f(2) = 7$
$f'(x) = 8x - 6$	$f'(x) = 2x$
$f'(x) = 6x - 3$	$f'(x) = 2 - 2x$
$f'(x) = 8x - 3$	$f'(2) = 10$
$f'(4) = 21$	$f'(1) = 5$
$f'(-1) = -2$	$f'(3) = -4$

C4 • Differentiating and integrating fractional and negative powers

Mathematical goals

To enable learners to:

- convert functions into an appropriate form for differentiating or integrating;
- differentiate negative and fractional powers of x;
- integrate negative and fractional powers of x.

Starting points

Learners should be able to differentiate and integrate polynomial functions and have some knowledge of fractional and negative indices.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Functions*;
- Card set B – *Functions in index form*;
- Card set C – *Differentiated functions*;
- Card set D – *Integrated functions*;
- Card set E – *Differentiated functions in square root and fractional form*;
- Card set F – *Integrated functions in square root and fractional form*.

Each set should be photocopied on to different coloured card.

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Use mini-whiteboards to briefly revise differentiation and integration of polynomials, fractional and negative indices. Ask questions such as:

For $y = x^4 - 3x^2 + 2x - 7$, find $\frac{dy}{dx}$.

$\int (x^3 + 4x^2 - 5x + 1)\, dx$

Write $\frac{1}{2}$ as a power of 2.

Write $\frac{1}{3^2}$ as a power of 3.

Write $\sqrt{x}$ as a power of x.

Write $\frac{1}{x^4}$ as a power of x.

How else can $\frac{1}{2x^3}$ be written?

It is likely that some learners will write the last one as $2x^{-3}$. If this happens, discuss this answer in detail, possibly demonstrating that it is not equivalent by substituting $x = 1$.

Leave any jottings on the board for learners to refer to as the session progresses.

Working in groups

Ask learners to work in pairs. Give each pair Card set A – *Functions*, Card set B – *Functions in index form*, Card set C – *Differentiated functions* and Card set D – *Integrated functions*. Ask learners to start with Set A and match them to Set B in order to prepare the functions for differentiating and integrating.

Next, ask them to match Sets C and D to Set B by applying the same principles to fractional and negative indices that they use for polynomial functions of x.

Learners may struggle to decide what happens to the constant (in this case the 2). Refer them back to the work done earlier in the session and left on the board.

If learners are not sure if their matchings are correct they can check their integration by differentiating and vice versa. Pairs of learners can also compare their matchings with other pairs and discuss any differences or problems until they come to a consensus.

Learners who complete the task quickly can be given Card set E – *Differentiated functions in square root and fractional form* and Card set F – *Integrated functions in square root and fractional form* to match with their integrals and differentials.

Reviewing and extending the learning

When all learners have matched sets A, B, C and D, ask each pair to explain one of their matchings to the rest of the group as a whole. Check understanding using mini-whiteboards and asking questions similar to those on the cards.

Ask learners to generalise their findings to find integrals and differentials such as:

$$\int \frac{a}{x^n}\,\mathrm{d}x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a}{x^n}\right) \qquad \int \sqrt[n]{x}\,\mathrm{d}x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt[n]{x}\right)$$

What learners might do next

Find stationary points and solve equations involving fractional and negative indices.

Find areas under a curve.

Further ideas

This approach could be used for differentiation and integration of any type of function.

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C4 Card set A – *Functions*

$y = 2\sqrt{x}$	$y = \dfrac{1}{2x^3}$
$y = \dfrac{2}{x^2}$	$y = \sqrt{x}$
$y = \dfrac{2}{\sqrt{x}}$	$y = \dfrac{1}{2\sqrt{x}}$
$y = \dfrac{1}{2x^2}$	$y = \dfrac{1}{\sqrt{x}}$
$y = \dfrac{2}{x^3}$	$y = \dfrac{1}{x^2}$

C4 Card set B – *Functions in index form*

$y = 2x^{-2}$	$y = 2x^{-\frac{1}{2}}$
$y = \frac{1}{2}x^{-2}$	$y = 2x^{\frac{1}{2}}$
$y = x^{\frac{1}{2}}$	$y = 2x^{-3}$
$y = x^{-2}$	$y = \frac{1}{2}x^{-\frac{1}{2}}$
$y = x^{-\frac{1}{2}}$	$y = \frac{1}{2}x^{-3}$

C4 Card set C – *Differentiated functions*

$\frac{dy}{dx} = -4x^{-3}$	$\frac{dy}{dx} = -\frac{1}{4}x^{-\frac{3}{2}}$
$\frac{dy}{dx} = -6x^{-4}$	$\frac{dy}{dx} = x^{-\frac{1}{2}}$
$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$	$\frac{dy}{dx} = -x^{-3}$
$\frac{dy}{dx} = -x^{-\frac{3}{2}}$	$\frac{dy}{dx} = -\frac{3}{2}x^{-4}$
$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$	$\frac{dy}{dx} = -2x^{-3}$

C4 Card set D – *Integrated functions*

$\int y\,dx = \frac{4}{3}x^{\frac{3}{2}} + c$	$\int y\,dx = -\frac{1}{2}x^{-1} + c$
$\int y\,dx = -x^{-2} + c$	$\int y\,dx = -\frac{1}{4}x^{-2} + c$
$\int y\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$	$\int y\,dx = -2x^{-1} + c$
$\int y\,dx = 2x^{\frac{1}{2}} + c$	$\int y\,dx = 4x^{\frac{1}{2}} + c$
$\int y\,dx = -x^{-1} + c$	$\int y\,dx = x^{\frac{1}{2}} + c$

C4 Card set E – ***Differentiated functions in square root and fractional form***

$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$	$\frac{dy}{dx} = -\frac{6}{x^4}$
$\frac{dy}{dx} = \frac{-3}{2x^4}$	$\frac{dy}{dx} = -\frac{2}{x^3}$
$\frac{dy}{dx} = -\frac{1}{4\sqrt{x^3}}$	$\frac{dy}{dx} = -\frac{4}{x^3}$
$\frac{dy}{dx} = -\frac{1}{\sqrt{x^3}}$	$\frac{dy}{dx} = -\frac{1}{x^3}$
$\frac{dy}{dx} = -\frac{1}{2\sqrt{x^3}}$	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

C4 Card set F – *Integrated functions in square root and fractional form*

$\int y\,dx = -\frac{2}{x} + c$	$\int y\,dx = -\frac{1}{x} + c$
$\int y\,dx = -\frac{1}{2x} + c$	$\int y\,dx = \sqrt{x} + c$
$\int y\,dx = \frac{2}{3}\sqrt{x^3} + c$	$\int y\,dx = \frac{4}{3}\sqrt{x^3} + c$
$\int y\,dx = 2\sqrt{x} + c$	$\int y\,dx = -\frac{1}{4x^2} + c$
$\int y\,dx = -\frac{1}{x^2} + c$	$\int y\,dx = 4\sqrt{x} + c$

C5 • Finding stationary points of cubic functions

Mathematical goals

To enable learners to:

- find the stationary points of a cubic function;
- determine the nature of these stationary points;

and to discuss and understand these processes.

Starting points

Learners should have some knowledge of:

- differentiation of polynomials;
- finding stationary points of a quadratic function;
- using $\frac{d^2y}{dx^2}$ or $f''(x)$ to determine their nature.

Materials required

For each learner you will need:

- mini-whiteboard;
- Sheet 1 – *Matching*.

For each small group of learners you will need:

- Card set A – *Stationary points*;
- large sheet of paper for making a poster;
- glue stick;
- felt tip pens.

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Use mini-whiteboards and whole group questioning to revise differentiation of polynomials and stationary points of a quadratic function by asking a range of questions such as:

If $y = 4x^2 + 5x - 1$ what is $\frac{dy}{dx}$?

If $y = x^3 - 8x^2 + 6x - 1$ what is $\frac{dy}{dx}$?

Find the x coordinate of the stationary point of $y = x^2 - 8x + 3$.

How do you know whether it is a maximum or a minimum?

Explain that, to find the stationary points on a cubic graph, exactly the same method is used. You may like to encourage learners to think about the likely shape of the graph.

Working in groups (1)

Ask learners to work in pairs.

Give out Card set A – *Stationary points* and ask learners to sort the cards into functions. Explain that the cards include three cubic functions and all the steps required to find their stationary points and to determine the nature of those stationary points. Learners have to sort the cards/steps into an appropriate order and stick them onto the poster paper in three columns, one for each function.

Learners who struggle can be given just one or two functions to sort.

Learners who find the task easy could be asked to find examples of cubic functions that have only one stationary point or no stationary points. You should encourage them to relate their answers to the derived quadratic functions.

Whole group discussion

When all learners have completed at least one function, discuss the order in which they have stuck down the steps. Ask learners to compare their orders with pairs sitting near them and to report any differences. Discuss how the orders can be checked and which orders are logically correct.

Working in groups (2)

Ask learners to write a comment beside each step of one of their functions, explaining its purpose. Finally, ask them to sketch a graph of the function, marking the stationary points.

Reviewing and extending the learning

Give individual learners Sheet 1 – *Matching* and ask them to match the equations to the graphs, justifying their choice as fully as they can. You may wish to ask them to match only two equations and graphs.

What learners might do next

Learners could be asked to explore and then sketch a given cubic function that factorises. This would connect work using the factor theorem with this session on stationary points.

Further ideas

This approach could be used to find stationary points of other types of functions (e.g. functions with fractional or negative indices).

This way of sorting the order of a solution can be used for solving multi-stage problems on any topic.

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C5 Card set A – *Stationary points* (page 1)

$y = x^3 - 4x^2 + 5x + 11$	$y = x^3 - x^2 - x + 5$
$y = x^3 - 7x^2 - 5x + 9$	$3x^2 - 8x + 5 = 0$
$\frac{d^2y}{dx^2} = 6x - 8$	$\frac{dy}{dx} = 3x^2 - 14x - 5$
$x = -\frac{1}{3}, \quad \frac{d^2y}{dx^2} = \ldots$	$x = \frac{5}{3}, \quad x = 1$
$\frac{dy}{dx} = 3x^2 - 2x - 1$	$\frac{d^2y}{dx^2} = 6x - 2$
$x = 1, \quad \frac{d^2y}{dx^2} = \ldots$	$x = -\frac{1}{3}, \quad x = 5$
$(3x + 1)(x - 5) = 0$	$x = 1, \quad \frac{d^2y}{dx^2} = \ldots$
$3x^2 - 14x - 5 = 0$	$3x^2 - 2x - 1 = 0$

C5 Card set A – *Stationary points* (page 2)

$(3x - 5)(x - 1) = 0$	$(3x + 1)(x - 1) = 0$
$x = 5, \quad \dfrac{d^2y}{dx^2} = \ldots$	$\dfrac{dy}{dx} = 3x^2 - 8x + 5$
$x = -\dfrac{1}{3}, \quad \dfrac{d^2y}{dx^2} = \ldots$	$x = -\dfrac{1}{3}, \quad x = 1$
$\dfrac{d^2y}{dx^2} = 6x - 14$	$x = \dfrac{5}{3}, \quad \dfrac{d^2y}{dx^2} = \ldots$
Maximum is at	**Minimum is at . . . .**
Minimum is at	**Maximum is at . . . .**
Maximum is at	**Minimum is at . . . .**

C5 Sheet 1 – *Matching*

Name:

Match these equations to their graph. Give as much evidence as you can to justify your matchings. The graphs are not drawn to scale.

Equation 1

$$y = x^3 + 2x^2 - 5x + 12$$

Equation 2

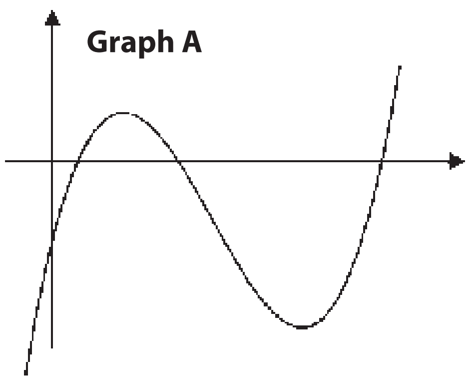
$$y = x^3 - 9x^2 + 22x - 10$$

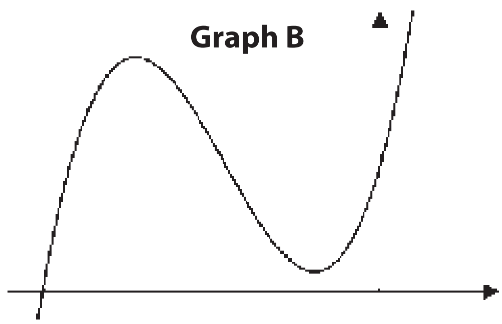
Equation 3

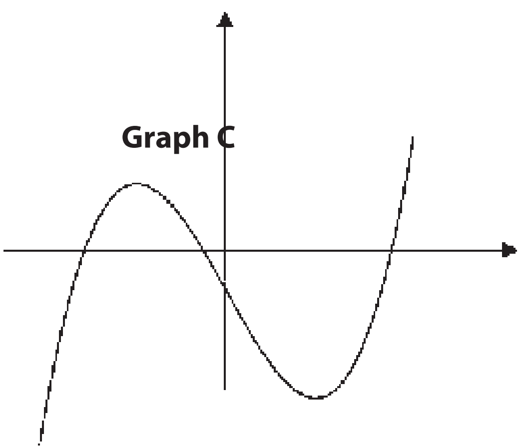
$$y = x^3 + 7x^2 + 15x + 12$$

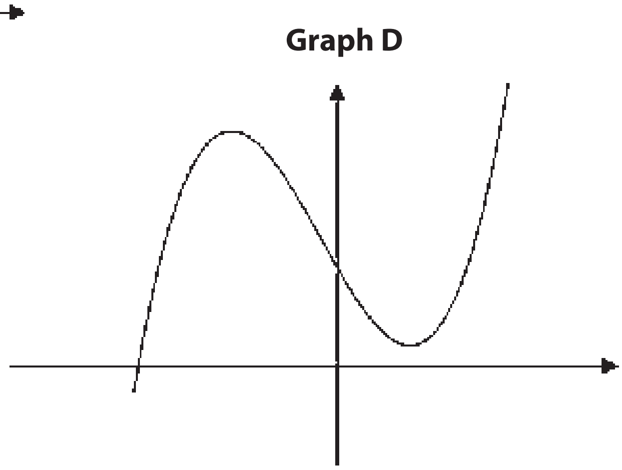
Equation 4

$$y = x^3 - 2x^2 - 13x - 10$$









O1 • Moving from Eulerian graphs to the route inspection (Chinese postman) problem

Mathematical goals

To enable learners to:

- distinguish, by drawing, between Eulerian graphs, semi-Eulerian graphs and graphs that are neither;
- distinguish, by using the order of the vertices, between Eulerian graphs, semi-Eulerian graphs and graphs that are neither;
- find strategies for solving the route inspection or 'Chinese postman' problem.

Starting points

Learners should have some knowledge of what is meant by a graph, a vertex and an edge in the context of decision mathematics.

Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Graphs*;
- Card set B – *Properties*;
- Card set C – *Completing the route*;
- glue stick;
- large sheet of paper for making a poster.

Time needed

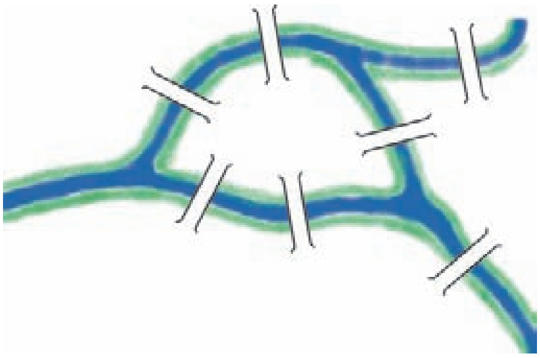
At least 1 hour 30 minutes.

Suggested approach

Beginning the session

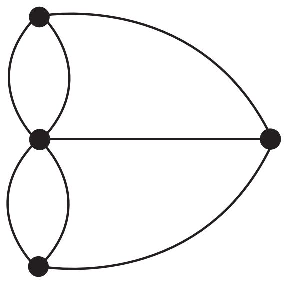
Tell the story of the bridges of Königsberg. Draw Figure 1 on the board.

Figure 1



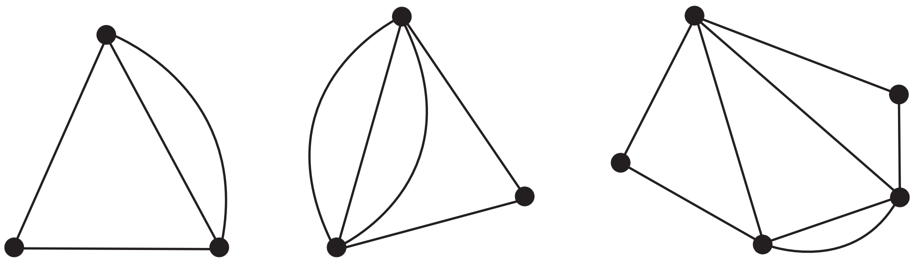
"In the town of Königsberg in the eighteenth century there was a river with an island and seven bridges, as shown in Figure 1. It is said that on Sunday afternoons the people of the town would try to walk across every bridge only once and end up back where they started from. Can it be done?"

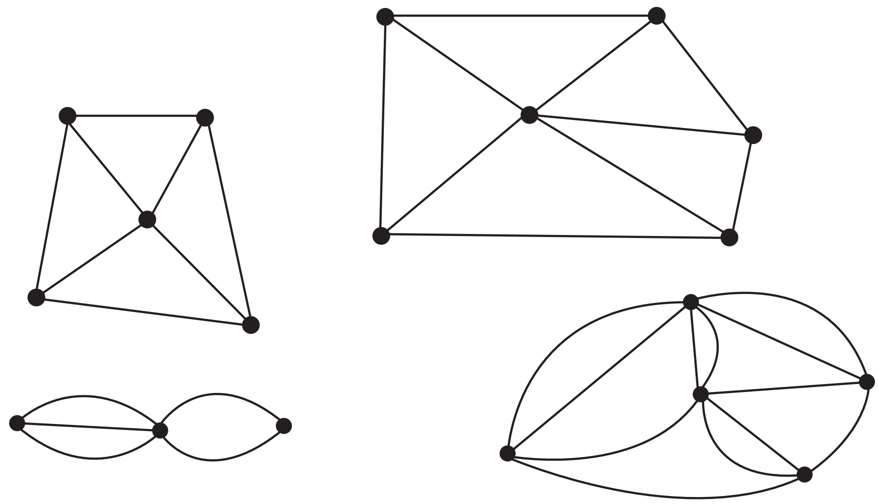
It is helpful to represent the problem as a graph. The vertices represent land and the edges represent the bridges.



Give out mini-whiteboards and ask learners to try to solve the problem for themselves. They should find that it is impossible. Tell them that Euler, a Swiss mathematician, eventually proved that it was impossible – but how did he do it?

Draw some simple graphs on the board, e.g.





Ask learners to decide which graphs can be drawn without going along any edge more than once and without lifting the pencil from the paper.

Next, ask them to divide the graphs that can be drawn in this way into two categories:

- those that start and finish at the same vertex;
- those that have to start and finish at different vertices.

You will need mini-whiteboards or lots of scrap paper so that learners can be encouraged to try different ways.

Whole group discussion

Come to a consensus about which graphs fit into each category. If there are any disagreements, ask learners to demonstrate on the board until a consensus is reached.

Explain that those that start and finish at the same vertex are called 'Eulerian graphs' and those that start and finish at different vertices are called 'semi-Eulerian graphs'. The remaining graphs are called 'non-Eulerian'. Also, explain that the number of edges leaving a vertex is called the 'order' of the vertex.

Ask learners to discuss, in small groups, what properties the vertices have to have in order for a graph to be assigned to each category. Help learners to reach the conclusion that all the vertices on Eulerian graphs have to be of even order, while two of the vertices on semi-Eulerian graphs have to be odd and the rest even. Encourage learners to explain why this must be so. Go back to the bridges of Königsberg and ask for an explanation as to why the townspeople's task is impossible.

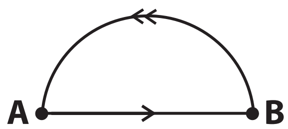
Working in groups

Ask learners to work in pairs. Give out a large sheet of paper and a glue stick to each pair. Ask them to divide the sheet into four. Give each pair Card set A – *Graphs* and ask them to stick one graph into each of the four sections. Then give out Card set B – *Properties* and Card set C – *Completing the route*. Ask them to stick an appropriate Properties card into each section. Learners should write out the reason as well as listing the odd vertices.

Explain the route inspection (Chinese postman) problem.

> Find a route that travels along all the edges. You may go down an edge more than once but you must keep the total distance travelled to a minimum.

You may need to remind learners that they should show any edges travelled more than once by adding additional edges to the diagram. Thus, travelling from A to B and then B to A would be shown as:



For each problem they should record what their solution is on one of the *Completing the route* cards and stick it on to the sheet next to its graph. At this stage they should be solving the problem using their own strategies.

Compare results and see which pair managed to obtain the shortest route for each graph. If none got the shortest possible on any of the graphs, challenge them to find a shorter one. Discuss strategies and come up with the rules for the Chinese postman algorithm.

Reviewing and extending the learning

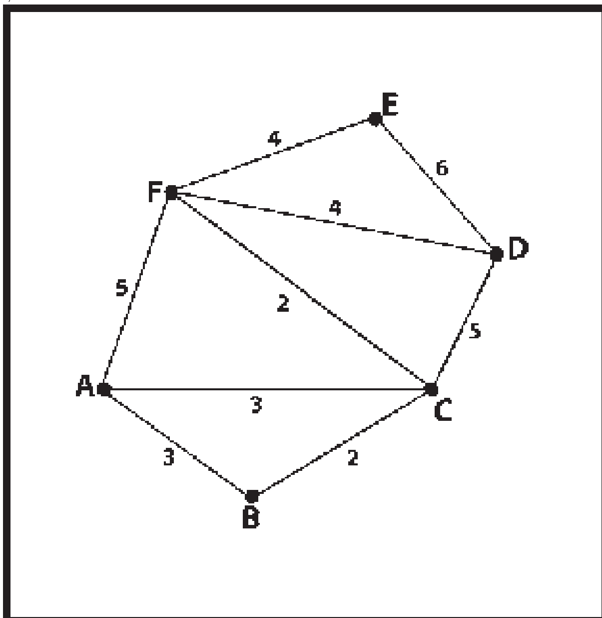
Ask each pair of learners to draw a graph of their own such that four of the vertices have an odd order. Ask them to write out the rules alongside a solution for their graph. When they have finished, they should give it to another pair for checking and comment.

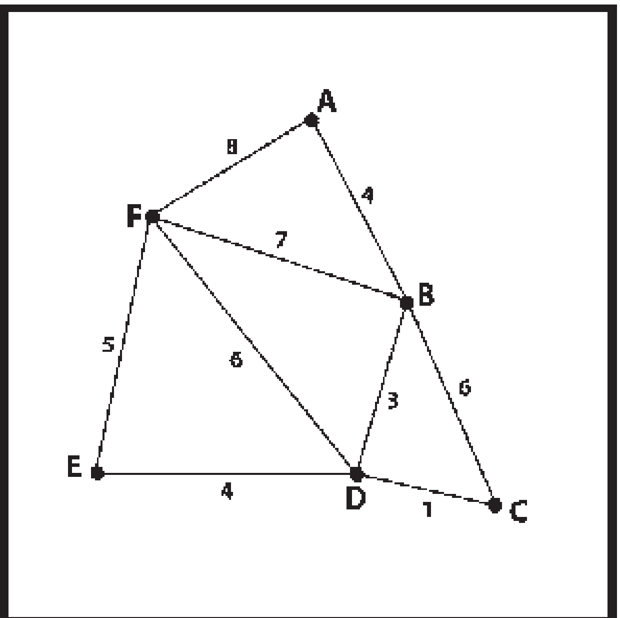
What learners might do next

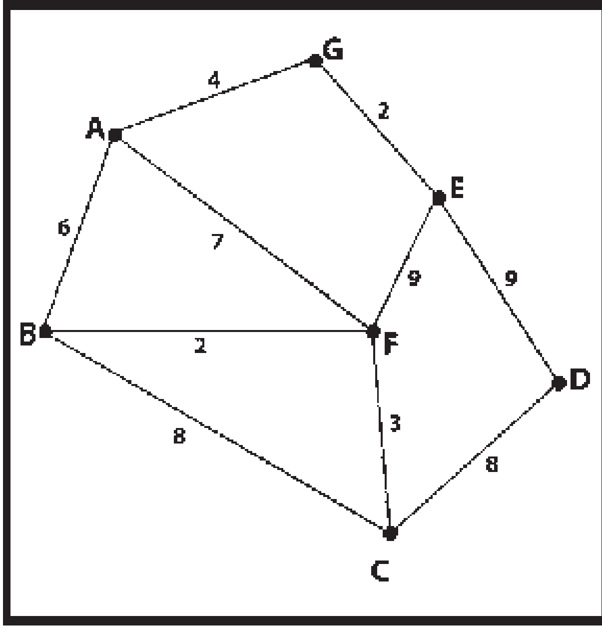
Explore how many possible pairings there are for 6, 8, 10 . . . n vertices that have an odd order.

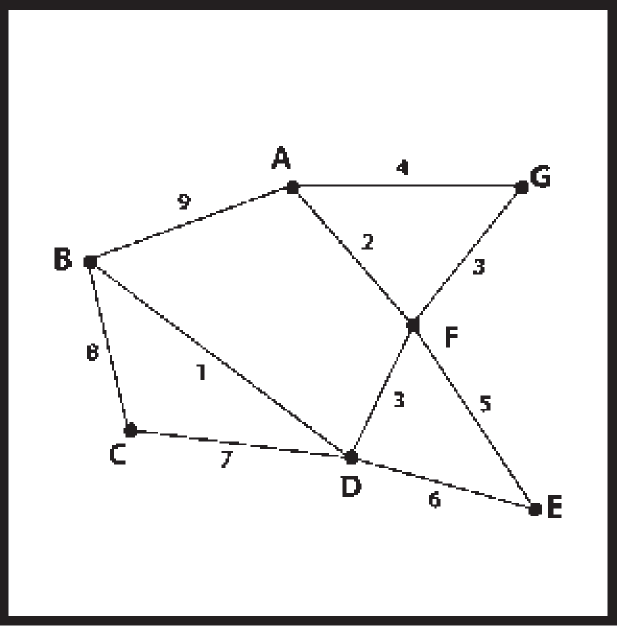
Investigate whether it is possible to have a graph that has an odd number of vertices that have an odd order.

O1 Card set A – *Graphs*









O1 Card set B – *Properties*

This is an Eulerian graph because… The odd vertices are:	This is a semi-Eulerian graph because… The odd vertices are:
This is a non-Eulerian graph because… The odd vertices are:	This is a semi-Eulerian graph because… The odd vertices are:

O1 Card set C – *Completing the route*

Extra edge(s) are: Possible route: Total length of route:	Extra edge(s) are: Possible route: Total length of route:
Extra edge(s) are: Possible route: Total length of route:	Extra edge(s) are: Possible route: Total length of route:

O2 • Exploring equations of motion

Mathematical goals

To encourage learners to:

- use past paper examination questions creatively.

To give learners practice in:

- using the equations of motion for constant acceleration.

To develop learners' ability to:

- generalise from specific situations of motion.

Starting points

Learners should have some knowledge and understanding of the equations of motion for constant acceleration.

Materials required

For each learner you will need:

- Sheet 1 – *Examination question*.

Time needed

At least 45 minutes.

Suggested approach

Beginning the session

Give each learner a copy of Sheet 1 – *Examination question*. Allow a maximum of five minutes for them to have a go at the question. Quickly go over the answers and ask learners to mark their own work.

Working in groups (1)

Ask learners to work in pairs to come up with a possible part (c) for the question.

Whole group discussion (1)

Write a range of questions that learners have come up with on the board and discuss how some of them could be tackled.

Alternatively, exchange part (c) questions between pairs of learners and ask them to solve them. Move round the room listening to their discussions and then, in whole group discussion, share any difficulties they have had in understanding or solving the question.

Working in groups (2)

Ask each pair of learners to come up with a possible 'What might happen to the boat next?' scenario.

Whole group discussion (2)

Ask each pair of learners to describe their scenario. Write a key phrase or a diagram on the board as a reminder of that scenario. Possible suggestions may include:

- boat goes over a waterfall;
- passenger jumps overboard;
- boat collides with another boat, or similar.

If the mechanics of a scenario are beyond what some of the learners have studied before, just have some general discussion about the effects of the scenario on the velocity and acceleration of the boat.

If the topic is due to be learned shortly, the scenario could be saved for that time or the topic could be introduced through the scenario.

Working in groups (3)

Choose a scenario that learners can investigate using the laws of motion, e.g. going over a waterfall. Ask learners, working in pairs, to come up with some relevant mechanics for the scenario.

Whole group discussion (3)

Share the mechanics of the scenario that learners have suggested and explore it further if appropriate.

Working in groups (4)

Ask pairs of learners to write a part (d) for the examination question, using this scenario and the mechanics that they have explored. Exchange part (d) questions between pairs of learners and ask each pair to complete the question that they have been given and return it to the authors for checking.

Other suggested scenarios can be used in a similar way.

Reviewing and extending learning

If not many learners completed the original examination question correctly, go back to it and ask learners to have another go. Then ask learners to generalise the original problem and solve it using variables rather than numerical values.

Ask each pair of learners to write an examination question together with its marking scheme. Give them as many constraints as you want, e.g. the question must involve a train or a ball and/or the equation $s = ut + \frac{1}{2}at^2$, and so on.

What learners might do next

Learners could develop the mechanics of the scenarios whose solution required more than just the laws of motion.

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O2 Sheet 1 – *Examination question*

A boat is travelling in a straight line down a river. On one particular stretch of the river the boat is moving with constant acceleration; its velocity increases from 10 m s^{-1} to 16 m s^{-1} in 4 seconds.

(a) Find how far the boat travels during this time.

(b) Find the acceleration of the boat.

O3 • Solving problems using Newton's second law

Mathematical goals

To help learners to:

- understand how Newton's second law can be applied to a range of different problems.

To encourage learners to:

- link the stages of solutions together;
- appreciate the purpose and relevance of each stage of a solution.

Starting points

Learners should have some knowledge of Newton's second law and how to resolve vectors.

Materials required

For each learner you will need:

- mini-whiteboard

For each small group of learners you will need:

- Sheet 1 – *Problem 1*;
- Sheet 2 – *Problem 2*;
- Sheet 3 – *Problem 3*;
- Sheet 4 – *Problem 4*;
- Sheet 5 – *Problem 5*.

Time needed

At least 1 hour.

Suggested approach

Beginning the session

Explain that learners will be given one part of each of five different problems. The task is to complete each problem by providing the missing parts.

Working in pairs

Ask learners to work in pairs. Give each pair a set of five problems (Sheets 1–5). Ask learners to suggest ways of completing the other four parts of each problem so that they fit with the one part that is given.

Whole group discussion

When learners have finished at least three of the problems, whole group discussion can revolve around the suggestions they have written and whether they are appropriate. For example, pairs can take it in turns to read out their suggested version of Problem 4. After each version, the rest of the learners can draw the force diagram appropriate to that problem and check it against the original suggestion from the learners. Mini-whiteboards could be used for this.

Some learners can read out their suggestions for Problems 2 and 3. After each suggestion is read out, discussion can revolve around whether it is appropriate for the diagram in Problem 2 or the resolved forces in Problem 3. Final solutions for these problems can also be checked against each other.

The assumptions can also be discussed and added to if appropriate.

Groups can exchange sheets to check whether the diagrams, components and solving fit together. Any discrepancies can be resolved through whole group discussion.

Reviewing and extending the learning

Ask pairs of learners to go back to Problem 1 in the light of the whole group discussions and make any adjustments that they feel are necessary. Exchange the solutions to Problem 1 between pairs and ask them to check that all parts are correct.

Solutions can be extended to more generalised situations.

What learners might do next

Link in with the laws of motion for constant acceleration by using the acceleration from Newton's 2nd law to determine speeds and/or distance and time.

Further ideas

This approach can be used whenever a solution has several stages that are linked together. It is particularly suitable for any mechanics problems that require several stages, e.g. a problem, a diagram, a solution and assumptions.

Not all the problems have to be given out together. Some could be held back to give out after discussion, or just a selection could be used for the session.

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O3 Sheet 1 – *Problem 1*

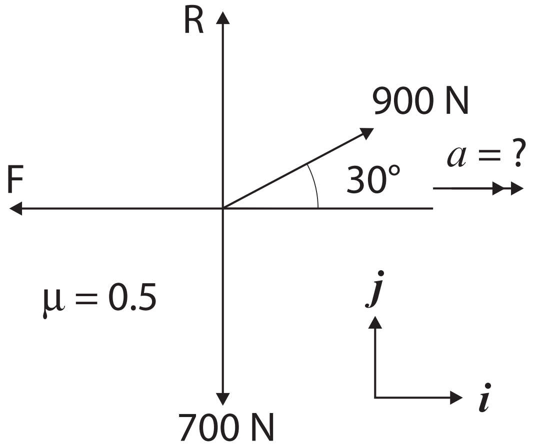
Problem

Peter and Andrew have been persuaded by Timothy to push him on his sledge. Timothy and his sledge have a combined mass of 50 kg. Timothy wants to go really fast but Peter and Andrew are feeling tired after lots of snowballing and making a big snowman. They reckon that they can only manage a horizontal push of 300 N between them. Timothy wants to know how much acceleration this will give him. If the coefficient of friction between the sledge and the ground is 0.4, what is the acceleration of the sledge?

Force diagram:	**Resolving into components:**
Solving:	
Assumptions:	

O3 Sheet 2 – *Problem 2*

Problem

Force diagram:	**Resolving into components:**
	
Solving:	
Assumptions:	

O3 Sheet 3 – *Problem 3*

Problem

Force diagram:	Resolving into components:
	$\boldsymbol{i}$: $100 \cos 40° + P \cos 60° - 25 = 80a$ $\boldsymbol{j}$: $100 \sin 40° - P \sin 60° = 0$

Solving:

Assumptions:

O3 Sheet 4 – *Problem 4*

Problem

Force diagram:	**Resolving into components:**

Solving:

$R = 600 - 500 \sin 50° = 217.0$ N

$F = 500 \cos 50° - 180 = 141.4$ N

$F = \mu R \Rightarrow \mu = F \div R = 141.4 \div 217.0 = 0.65$

Assumptions:

O3 Sheet 5 – *Problem 5*

Problem

Force diagram:	**Resolving into components:**

Solving:

Assumptions:

The acceleration of the lift remains constant at 5 m s^{-2}.

The man remains still and in contact with the floor of the lift.